PHD

Duality Methods for Barrier-type Solutions to the Skorokhod Embedding Problem

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Award date:
2018

Awarding institution:
University of Bath

Duality Methods for Barrier-type Solutions to the Skorokhod Embedding Problem

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for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

June 2018

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DECLARATION OF AUTHORSHIP

I am the author of this thesis, and the work described therein was carried out by myself personally. Chapters 2, 3, and 4 were carried out in collaboration with my supervisor Dr Alexander Cox. Chapter 5 was carried out in collaboration with Dr Alexander Cox and Dr Jiajie Wang.

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SUMMARY

The Skorokhod embedding problem is to find a stopping time of a Brownian motion, W, for which the stopped process has a given distribution. The Root, Rost, and cave embedding solutions to the problem can be seen as the first hitting time for (W_t, t) of regions known as barriers, inverse barriers, and cave barriers, respectively. In this thesis we present three ways of approaching the embedding problem, and apply the methods to these barrier-type solutions. Specifically, we consider infinite dimensional linear optimisation problems in both discrete and continuous time, and we also reformulate into an optimisation constrained by backwards stochastic differential equations and then solve using techniques from stochastic optimal control.

For certain financial derivatives it is well known that there is an optimal Skorokhod embedding problem which corresponds to finding a model-independent upper bound on the price of the contingent claim. With this application in mind, the embedding problem has the dual problem of finding the minimal cost of a superhedging portfolio for the option. The methods developed in this thesis enable us to explore the relation between the primal and dual problems, and, in the applications above, find dual optimisers. We also introduce a new barrier-type embedding, known as a K-cave embedding, which has the property of maximising the price of a European call option on a leveraged exchange traded fund. For the cave and K-cave embeddings the attainment of an optimal superhedging strategy is needed to find the optimal barriers. Unlike in the cases of Root and Rost, there are not unique cave, or K-cave barriers which embed a given distribution and in this way these are the first examples of embeddings which are not uniquely determined by their geometric structure.

ACKNOWLEDGEMENTS

First, my sincere thanks to my supervisor, Alexander M. G. Cox. I am incredibly grateful for the time and support he has given me whilst I worked on the exceptionally interesting problems that he introduced me to.

A big thank you to everyone who has helped make my life fun during my PhD, from Bath and further afield. In particular I would like to thank Fred and Ray for all the beers and a lot of fun, Matt for the many years of chats and advice, and Groom and the guys at home for the constant laughs. Special thanks go to Jack for the daily chats that kept me going, whatever life threw at us both.

I would also like to express my gratitude to Mr and Mrs Barratt, who have always supported me and taught me to be proud of myself.

A very important thank you to Jessica for always being there for me and always knowing what to say, you help me every single day.

Finally, to my parents and family, thank you for always believing in me and inspiring me, your encouragement made this thesis possible.

Contents

List of Figures

Chapter 1

Introduction

Many years since the work of Brown, Einstein, and Wiener, the Brownian motion remains at the forefront of modern probability theory, and is a fundamental object in the wider world of pure and applied mathematics.

The focus of this thesis is one particular problem related to the study of Brownian motion, the *Skorokhod embedding problem*. Skorokhod [1961, 1965] posed the original question: suppose W is a one-dimensional Brownian motion and μ is a distribution on $\mathbb{R}$. When can we find a stopping time τ such that W_τ has distribution μ?

1.1 Development of the Skorokhod Embedding Problem

The first solution of the problem was given by Skorokhod alongside its statement in Skorokhod [1965]. Assuming that the measure μ is centred, Skorokhod constructed random variables X and Y, independent of W such that the stopping time

$$\tau_S := \inf\{t \geq 0 : \; W_t \notin [X, Y]\}$$

satisfies $W_{\tau_S} \sim \mu$. This solution has some desirable properties, which we highlight by means of comparison with another, simpler, solution attributed to Doob.

If F is the cumulative distribution of μ, and Φ the cumulative distribution of a $\mathcal{N}(0, 1)$ random variable, then the stopping time

$$\tau_D := \inf\{t \geq 1 : \; F(W_t) = \Phi(W_1)\}$$

also solves the Skorokhod embedding problem. This is easy to check since

$$\begin{aligned}\mathbb{P}\left(W_{\tau_D} \leq x\right) &= \mathbb{P}\left(F^{-1}(\Phi(W_1)) \leq x\right) \\ &= \mathbb{P}\left(W_1 \leq \Phi^{-1}(F(x))\right) \\ &= \Phi(\Phi^{-1}(F(x))) \\ &= F(x).\end{aligned}$$

Then, for a centred distribution μ, the solution is not unique. The natural question is then: which of these solutions is 'better'? Or more generally, what are the desirable properties of solutions?

If μ is centred and has finite second moment, then $\mathbb{E}\left[\tau_S\right] = \mathbb{E}\left[W_{\tau_s}^2\right] < \infty$, whereas $\mathbb{E}\left[\tau_D\right] = \infty$ (unless $\mu = \mathcal{N}(0,1)$). It is usually desirable to ensure that $\mathbb{E}\left[\tau\right]$ is 'small', for example if we are required to move to a framework with a finite time horizon as in optimal stopping theory, so in this situation we favour τ_S. One possibility is to restrict the problem to finding stopping times τ with finite expectation, which is equivalent to requiring that $(W_{t\wedge\tau})_{t\geq 0}$ is a square-integrable martingale. Many solutions to the embedding problem make the assumption that μ is centred and $(W_{t\wedge\tau})_{t\geq 0}$ is a uniformly integrable martingale, and embeddings τ satisfying this property are known as UI embeddings.

In this thesis, the form of the problem we work with is: given a Brownian motion W and a centered probability distribution μ on the real line which has finite second moment, the Skorokhod embedding problem is to find a (possibly randomised) stopping time τ such that

$$W_\tau \text{ has law } \mu \text{ and } (W_{t\wedge\tau})_{t\geq 0} \text{ is UI.} \tag{SEP}$$

There are multiple generalisations of this problem, an obvious example being the extension to more general processes. Rost [1971] shows that there is a (possibly randomised) stopping time embedding the distribution μ into a Markov process X with $X_0 \sim \nu$ if and only if

$$\nu U^X \geq \mu U^X,$$

where U^X is the potential kernel of X. We discuss potential theory further in Section 1.3.

Another generalisation is to consider the multi-marginal Skorokhod embedding problem: given measures $\mu_0, \ldots, \mu_n$ of finite variance, and a Brownian motion W with

$W_0 \sim \mu_0$, construct stopping times $\tau_1 \leq \ldots \leq \tau_n$ such that

$$W_{\tau_i} \sim \mu_i \text{ and } \mathbb{E}\left[\tau_i\right] < \infty \text{ for all } 1 \leq i \leq n.$$

Similarly to the above ordering on the potential kernels, such an embedding exists if and only if the measures are increasing in convex order, written $\mu_0 \preceq_c \ldots \preceq_c \mu_n$, meaning that for any convex $\varphi : \mathbb{R} \to \mathbb{R}$

$$\int \varphi(x)\mu_0(\mathrm{d}x) \leq \ldots \leq \int \varphi(x)\mu_n(dx).$$

The multi-marginal problem has been considered in, for example, Madan et al. [2002], Beiglböck et al. [2017a], and Cox et al. [2018].

1.1.1 Solutions

The Skorokhod embedding problem has been studied continuously since its introduction, and this is largely due to the wide range of applications of the problem. New uses continue to be found, and with these come new approaches and solutions to the problem. Embeddings can be used in proofs of invariance principles such as Donsker's theorem, in the field of robust, or model-independent, finance, and more recently the embedding problem has been linked with the field of optimal transport. The financial relevance and the relation to optimal transport will be explained fully in Section 1.3, but now we explore a handful of the many elegant solutions to the original problem.

We have seen that in general there can be multiple stopping times embedding a given distribution, but we have only encountered one UI embedding, the solution of Skorokhod himself. There are numerous examples of UI embeddings, and we again need some measure of which solutions have 'nice' properties. From the above application it appears natural to choose an embedding which has some maximal or minimal properties, and this is how many embeddings in the literature have been constructed. Here we list a small sample of these embeddings.

- Root [1969]: Root's solution takes the form of a hitting time for (W_t, t) of a region known as a barrier. This solution, and the similar Rost solution, will be discussed in more detail in Section 1.3.
- Azéma and Yor [1979]: this solution is the entrance time into a region of $(\overline{W}_t, W_t)$, where $\overline{W}_t = \sup_{s\leq t} W_s$ is the maximum process. The stopping time maximises

the law of the maximum among the UI embeddings.

- Vallois [1983]: the Vallois solution maximises the distribution of the local time at zero and is a hitting time for (W_t, L_t) of some region, where L is the local time of W at 0.

There are many more known embeddings, for example Chacon and Walsh [1976], Bass [1983], Perkins [1986], Obłój and Yor [2004], and also works offering different approaches to these embeddings and developing them further. We refer the reader to Obłój [2004] for a comprehensive list of all solutions of the embedding problem known at the time, and also for further details and applications.

1.2 Outline

The majority of this thesis is concerned with the barrier-type solutions of the Skorokhod embedding problem. We give alternative ways of approaching the embedding problem: as a linear optimisation problem over spaces of measures in both continuous and discrete time, and as a stochastic optimal control problem. We apply these methods to the Root, Rost, cave, and K-cave embeddings.

1.2.1 Chapter 1.3: Preliminaries

In this preliminary section we give details on the relation between the Skorokhod embedding problem and model-independent finance, in particular the robust hedging of options and the corresponding dual problem. We introduce the solutions of Root and Rost, and the related cave solution, and explain their remarkable optimality properties. We will also explore the relationship between the embedding problem and martingale optimal transport, and look at the recent advances made in the area due to the link between the two fields.

1.2.2 Chapter 2: Discretisation of Optimal Skorokhod Embedding Problems

In this chapter we take a new approach to the optimal embedding problem by discretising our domain and considering an optimal stopping problem for a simple symmetric random walk Y on a grid $(x_j)_j$. This can be written as an infinite linear programming problem which has a well-defined dual. The variables are a sequence $(p_{j,t})_{j,t}$ corresponding to a stopping time σ where $p_{j,t} = \mathbb{P}(Y_t = x_j, \sigma > t)$ are continuation probabilities of the random walk, and we restrict these to a weighted l^p space in order to prove a strong duality result. This result states that there is no duality gap and that dual optimisers are obtained.

To make use of these results in the continuous-time optimisation we need to recover the original problem as a limit of discrete problems. When the Root, Rost, cave, and K-cave cases are considered we prove that there is an optimal stopping time for the random walk which corresponds to a discrete version of the appropriate stopping region (i.e. a discrete Root barrier, Rost inverse barrier etc.). We show that these regions, and their corresponding stopping times, converge to their continuous counterparts, and that the limiting stopping time is a solution of the optimal Skorokhod embedding problem.

The discrete setup allows us to prove properties of the optimisers more easily than in continuous time, and then prove that these properties hold in the limit. In particular we prove that the discrete problem permits dual attainment in a certain weighted space, and we use these ideas in Chapter 3 to find dual optimisers in certain continuous time problems, a problem which has been shown to be extremely non-trivial in general. Furthermore, the discretised problem can be easily reformulated to account for arbitrary starting measures and various constraints on the local time of the target measure.

1.2.3 Chapter 3: The K-cave Embedding

In this chapter we introduce the problem of maximising the expected payoff of a European call option on a financial product known as a leveraged exchange traded fund. Using the ideas of Section 1.3 this leads to an optimal Skorokhod embedding problem.

The first result in this chapter proves the existence of a solution to this problem which is the hitting time for (W_t, t) of a region called a K-cave barrier. The solution takes the form

$$\tau := \inf \{t \geq 0 : \, W_t \notin (l(W_t), r(W_t))\},$$

where $l : \mathbb{R} \to \overline{\mathbb{R}}$ and $r : \mathbb{R} \to \overline{\mathbb{R}}$ are upper and lower semi-continuous respectively and $l(x) \leq K(x) \leq r(x)$ for all x, where $K : \mathbb{R} \to \mathbb{R}$ is a given curve. The regions $\overline{\mathcal{R}} := \{(x,t) : \, t \leq l(x)\}$ and $\underline{\mathcal{R}} := \{(x,t) : \, r(x) \leq t\}$ are stopping regions corresponding to the solutions of Rost and Root respectively.

Unlike the solutions of Root and Rost, there is in general not a unique K-cave barrier embedding the correct distribution. We give an example of this non-uniqueness and explain why this makes this solution, and the related cave embedding solution, the first examples of embeddings which are not uniquely determined by their geometric structure. We then require an additional condition in order to choose the optimising embedding. The condition is motivated using PDE methods, and we prove the sufficiency of the condition using a probabilistic approach. In doing this we introduce the dual problem of finding the minimal initial cost of a model-independent superhedging portfolio. We give a feasible superhedging portfolio and show that this particular portfolio is optimal if and only if our condition holds. This condition is the same in the cases of the cave and the K-cave barrier.

To prove that our superhedge is indeed optimal, and thus the proposed condition is indeed necessary, we apply the discrete framework of Chapter 2 to the K-cave embedding problem. We show that the limit of the discrete dual optimisers is our suggested superhedging portfolio, proving its optimality, and are also able to deduce further features of the optimal K-cave barriers due to the strength of the discrete approach.

1.2.4 Chapter 4: Continuous-Time Optimisation Results

The goal of this chapter is to formulate, and then solve, a continuous-time equivalent of the linear programming problem established in Chapter 2. For any stopping time τ, the process $(W_{t\wedge\tau}, t\wedge\tau)_{t\geq 0}$ has a continuation measure p on $\mathbb{R}\times\mathbb{R}^+$, so that $p(A) = \int_t \mathbb{P}\left((W_t,t)\in A,\ t<\tau\right)\mathrm{d}t$. For 'small' A, this roughly corresponds to the probability that a path enters the set A but doesn't stop there. Note that any such measure will be dominated by the usual Brownian transition density, and will therefore have a density with respect to Lebesgue. We can then rewrite the optimal Skorokhod embedding problem as a deterministic optimisation problem over some function space.

Every continuation measure has a corresponding stopping measure, q say, however we do not in general expect the stopping measure to have a density. For example, the Root embedding of the $\mathcal{N}(0,1)$ measure is simply the stopping time $\tau\equiv 1$. The stopping distribution is therefore supported on $\{t=1\}$, so corresponds to some Dirac measure which does not have a density. The two measures are linked through the relationship $\frac{1}{2}p_{xx} - p_t = q$, where we interpret the derivatives in a distributional sense, and we also have other conditions on p and q such as non-negativity. We consider two primal problems, optimising over the sets

$$\mathsf{X} := \left\{ p : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} :\ p \in L^\infty,\ q(x,t) = \left(\frac{1}{2}p_{xx} - p_t\right)(x,t) \text{ exists weakly, } q \in L^\infty \right\}$$
$$\tilde{\mathsf{X}} := \left\{ p : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} :\ p \in L^\infty,\ q = \frac{1}{2}p_{xx} - p_t \text{ exists as a measure, } \|q\|_{TV} < \infty \right\}.$$

The set X proves easier to work with and so we prove duality results in this setting and later show that the two problems are equivalent. As in the discrete framework of Chapter 2, we also need to enforce some exponential decay of p and q in order to prove duality. Using various results from functional analysis we conclude a duality result which transfers back to the optimal Skorokhod embedding problem, ultimately providing an alternative way to formulate the problem when certain payoffs are considered.

1.2.5 Chapter 5: Optimal Skorokhod Embeddings as Stochastic Optimal Control Problems

As in the previous chapters, here we provide an alternative approach to an optimal embedding problem. We note that for any randomised stopping time τ we can define an increasing process R on $[0,1]$ by $R_t := \mathbb{E}\left[\mathbf{1}\{\tau \le t\} \middle| \mathcal{F}_t^W\right]$, where $\mathcal{F}^W$ is the natural filtration of W. Then for any suitable F, our payoff is $\mathbb{E}\left[F(W_\tau, \tau)\right] = X_0$, where

$$X_t := -\int_t^T R_s \mathcal{L}F(W_s, s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s + F(W_T, T)$$

for $0 \le t \le T$ and some Z. Note that we have a finite time horizon T since we wish to produce an optimal control problem in which such constraints are required.

To ensure that the stopping time embeds a measure smaller than μ in convex order, we introduce processes

$$Y_t(z) = w_\mu(z) - \int_0^t (1 - R_s)\mathrm{d}L_s^z + \int_0^t \xi_s(z)\mathrm{d}W_s,$$

for $0 \le t \le T$ and every $z \in \operatorname{supp}(\mu)$, and choose ξ to ensure that $Y_t(z) \ge 0$ for all t and z. Here w_μ is related to the potential function of μ which will be defined in Section 1.3.

The optimal Skorokhod embedding problem then has an equivalent BSDE formulation where we maximise over increasing processes $(R_t)_{t\ge 0}$ in the above. We then use R_t (or

$1 - R_t$) as a control process and include the constraints $Y_t(z) \geq 0$ into the objective function through the method of Lagrange multipliers to construct a min-max problem involving a classical stochastic optimal control problem. Tools such as the stochastic maximum principle can then be used to determine the optimal control, and we find that the corresponding stopping time is of the form of e.g. Root and Rost. The optimisation over the Lagrange multipliers ensures that the stopping time embeds the correct distribution, and so we fully recover the optimal embedding problem.

1.3 Preliminaries

1.3.1 Model-Independent Finance

Suppose we have some contingent claim on an underlying asset $(S_t)_{t\geq 0}$ which pays out $F(S_T, T)$ at time T. The traditional method of finding the price of such a contract, based on the ideas of Black and Scholes [1973], is to assume the existence of a risk-neutral measure and find the discounted expected payoff under this measure, so the time 0 price is

$$V(0) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-rT} F(S_T, T)\right],$$

for a fixed interest rate r.

Even if we know the law of S_T under a measure $\mathbb{Q}$, the choice of the risk-neutral measure exposes us to model risk: how do we choose $\mathbb{Q}$ in order to accurately capture the real world behaviour of the asset? One possible solution is to consider only those $\mathbb{Q}$ which are consistent with some observable prices in the markets. Breeden and Litzenberger [1978] show in the following result that the marginal distribution of the underlying at time T can be deduced from the prices of European call options with maturity T.

Lemma 1.1. *Suppose that European call options with maturity T are traded in the market at any strike $K \in (0, \infty)$. Let us further assume that their prices are computed as the discounted expected payoff under the probability measure $\mathbb{Q}$, that is, for any $K \in (0, \infty)$,*

$$C(K) = \mathrm{e}^{-rT} \mathbb{E}^{\mathbb{Q}}\left[(S_T - K)_+\right].$$

Then we have

$$\mathbb{Q}\left(S_T > K\right) = \mathrm{e}^{rt} \left| \frac{\partial C}{\partial K}(K+) \right|,$$

and under the assumption that C is twice differentiable

$$\mathbb{Q}(S_T \in \mathrm{d}K) = \mathrm{e}^{rT} \frac{\partial^2 C}{\partial K^2}(K)$$

has to hold.

In general there will be multiple $\mathbb{Q}$ that give the correct observed call option prices, but as a consequence of the above, we can determine the price of any derivative whose payoff depends only on T and S_T given the market observed call option values. If we wish to consider path-dependent payoffs then the marginal distribution at T is no longer sufficient to provide a unique value as each of the feasible measures $\mathbb{Q}$ may give a different valuation. Instead of looking for a single price, we can search for upper and lower bounds on this range of prices, and in particular we look for extremal models that give tight bounds. Consider the problem of maximising the price of an option with path dependent payoff $F(S_T, \langle S \rangle_T)$. Suppose that we observe call option prices and the Breeden and Litzenberger formula implies that $S_T \sim \mu$, for some μ, under feasible models. We therefore wish to find

$$\sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}} \left[F\left(S_T, \langle S \rangle_T\right) \right] \text{ over } \mathbb{Q} \text{ such that } S_T \overset{\mathbb{Q}}{\sim} \mu.$$

If $(S_t)_{t \geq 0}$ is a continuous local martingale, then we can use Dambis-Dubins-Schwarz to time change the process to become a Brownian motion, W, and $\langle S \rangle_T$ becomes a stopping time for the Brownian motion. We now have the optimal Skorokhod embedding problem

$$\sup_{\tau} \mathbb{E}\left[F\left(W_\tau, \tau\right) \right] \text{ over solutions to (SEP).} \tag{OptSEP}$$

Importantly we can also go in the other direction: if τ is such that $W_\tau \sim \mu$, then we can recover a martingale

$$M_t = W_{\frac{t}{T-t} \wedge \tau}$$

such that $M_T \sim \mu$.

These techniques were first used in Hobson [1998] to construct model-independent upper and lower bounds on the prices of lookback options. The approach has since been used in a number of hedging problems, including Brown et al. [2001], Hobson and Pedersen [2002], Davis and Hobson [2007], Cox and Obłój [2011a,b]. We refer the reader to Hobson [2011] for a survey of solutions of the Skorokhod embedding problem with applications to model-independent finance.

In principle, if sufficiently tight bounds on the price of a certain derivative can be found then arbitrage opportunities in the markets could be detected, but for this to be practically useful we require these bounds to be be attainable by some hedging portfolio. The primal problem of maximising the payoff of a given derivative has the corresponding dual problem of constructing a semi-static superhedging portfolio with minimal cost. A semi-static portfolio comprises a static position in vanilla options bought at initiation, and also a dynamic delta strategy. Under mild conditions it has been shown that there is no duality gap, see for example Beiglböck et al. [2013], Dolinsky and Soner [2014], Galichon et al. [2014].

It is shown in Beiglböck et al. [2013] that in general the problem does not always admit dual attainment, and so dual optimisers have, to the best of our knowledge, always been constructed case-by-case, for example in Hobson [1998], Cox and Obłój [2011a,b], Hobson and Klimmek [2012], Hobson and Neuberger [2012], and Cox and Wang [2013a,b].

1.3.2 Martingale Optimal Transport and the Monotonicity Principle

As mentioned previously, as new applications and motivations of the Skorokhod embedding problem emerge, new methods applicable to the problem evolve, and these techniques can then be used with earlier applications in mind. One such related topic from recent literature is martingale optimal transport, and techniques from classical optimal transport have given useful insights into the martingale counterpart and therefore the Skorokhod embedding problem. In this section we give a very short overview of martingale optimal transport, focusing on the results most useful in this thesis. For more information on classical optimal transport, see Villani [2009] and Ambrosio and Gigli [2013], and for more on martingale optimal transport see Beiglböck et al. [2013], Galichon et al. [2014], Dolinsky and Soner [2014], Beiglböck and Juillet [2016], Beiglböck et al. [2017c,d].

The original optimal transport problem of Monge [1781] was to move soil in a pile of distribution ν into a hole represented by distribution μ using an optimal bijection which minimises some cost function. The problem was generalised in Kantorovich [1942, 1948] to consider probability measures whose marginals agree with ν and μ.

More precisely, for probability measures ν, μ on $\mathbb{R}$, a Monge-Kantorovich transport from ν to μ is a probability measure π on $\mathbb{R}^2$ whose marginals are ν and μ. If (X, Y) is the identity map on $\mathbb{R}^2$, then we can write this as $X \overset{\pi}{\sim} \nu$ and $Y \overset{\pi}{\sim} \mu$. Denote the set of these probability measures by $\Pi(\nu, \mu)$, and note that this is non-empty since $\nu \otimes \mu \in \Pi(\nu, \mu)$. Take a upper-semicontinuous cost function $c : \mathbb{R}^2 \to \mathbb{R}$ such that $c(x, y) \geq a(x) + b(y)$ for some $a \in L^1(\nu)$ and $b \in L^1(\mu)$. Then the optimal transport problem is to find

$$\sup_{\pi \in \Pi(\nu, \mu)} \mathbb{E}^{\pi}\left[c(X, Y)\right].$$

Typically the infimum is considered in classical optimal transport, so we can consider c as the negative of some cost function.

The corresponding dual problem is

$$\inf_{\varphi, \psi} \nu(\varphi) + \mu(\psi), \qquad \text{subject to } \varphi(x) + \psi(y) \geq c(x, y),\ \forall (x, y) \in \mathbb{R}^2.$$

Kellerer [1984] shows that there is no duality gap, and that dual optimisers exist provided the optimal value is finite. Furthermore, there is a set $\Gamma \subseteq \mathbb{R}^2$ such that any $\pi \in \Pi(\nu, \mu)$ is optimal for the primal problem if and only if π is concentrated on Γ. This set Γ is c-cyclically monotone, meaning that for any set of points $(x_1, y_1), \ldots, (x_n, y_n) \in \Gamma$ we have

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{i+1}), \tag{1.1}$$

where $y_{n+1} = y_1$.

In the case of martingale optimal transport, we let

$$\mathcal{M}(\nu, \mu) := \left\{\pi \in \Pi(\nu, \mu) :\ \mathbb{E}^{\pi}\left[Y | X\right] = X,\ \pi - a.s.\right\},$$

and consider

$$\sup_{\pi \in \mathcal{M}(\nu, \mu)} \mathbb{E}^{\pi}\left[c(X, Y)\right].$$

In this case, ν can be thought of as the law of some martingale at time 0, and μ its law at some time T, and then we can see how this recovers a problem of the form $\sup_{\mathbb{Q}} \mathbb{E}^{\mathbb{Q}}\left[F(S_T)\right]$ from model-independent finance. A result of Strassen [1965] says that $\mathcal{M}(\nu, \mu)$ is non-empty if and only if ν, μ are in convex order, i.e. $\nu \preceq_c \mu$. The extra martingale constraint gives a further dual variable, h, and since $\mathbb{E}^{\pi}\left[Y | X\right] = X$ is equivalent to $\mathbb{E}^{\pi}\left[h(X)(Y - X)\right] = 0$ our dual problem becomes,

$$\inf_{\varphi, \psi, h} \nu(\varphi) + \mu(\psi),$$

over functions (φ, ψ, h) such that

$$\varphi(x) + \psi(y) + h(x)(y - x) \geq c(x, y), \quad \forall (x, y) \in \mathbb{R}^2. \tag{1.2}$$

Returning to the financial setting, the dual functions correspond to a superhedging strategy where φ and ψ are achieved through trading options, and we buy $h(x)$ stocks at time 0. It is shown in Beiglböck et al. [2013] that if c is upper-semicontinuous and satisfies a linear growth property then there is no duality gap, but the dual problem may not admit optimisers even in the case of well-behaved c. If we require (1.2) to hold only in a quasi-sure sense, then Beiglböck et al. [2017d] shows that there is no duality gap *and* dual optimisers are attained when the optimal value is finite.

The cases of path-dependent payoffs are considered in Dolinsky and Soner [2014] and Beiglböck et al. [2017b], and in the latter paper a monotonicity principle is established which allows the authors to prove, using a unified approach, the existence of all solutions to the Skorokhod embedding problem that have an optimality property. The monotonicity principle will be an important tool in this thesis and we sketch it in the following section.

The Monotonicity Principle and Stop-Go Pairs

We wish to construct a stopping rule for a set of paths so that the stopped paths maximise some payoff. As in the case of c-cyclical monotonicity we take a stopping rule and consider perturbing it. Consider a stopped path (g, t) and a path that is not yet stopped (f, s), where $f(s) = g(t)$. We imagine stopping (f, s) at time s and creating a continuation of (g, t) by transferring all paths which extend (f, s) onto (g, t). If this improves the value of the quantity we are optimising, then we have contradicted the optimality of the stopping region. In this case we call $((f, s), (g, t))$ a stop-go pair, and we denote the set of stop-go pairs by SG. This then can be extended by a second optimality problem in order to sort the pairs that see exactly the same value of the optimality problem when mass is transferred onto the stopped path.

As in Beiglböck et al. [2017b] we work on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_t)$ which is rich enough to support a Brownian motion, W, and a uniformly distributed $\mathcal{F}_0$-random variable. Formally, Beiglböck et al. [2017b] considers $\mathsf{S} = \{(f, s) : f : [0, s] \to \mathbb{R} \text{ is continuous}, f(0) = 0\}$ and a Borel function $\gamma : \mathsf{S} \to \mathbb{R}$, so $\gamma_t = \gamma((W_s)_{s \leq t}, t)$ is an optional stochastic process. Our problem is to find the maximiser of

$$P_\gamma := \sup\{\mathbb{E}[\gamma_\tau] : \tau \text{ solves (SEP)}\}. \tag{1.3}$$

We set

$$(\gamma^{\oplus(f,s)})_u := \gamma(f \oplus W, s+u),$$

and then (f,g) is a stop-go pair, $(f,g) \in \mathsf{SG}$, if for every stopping time σ such that $0 < \mathbb{E}[\sigma] < \infty$,

$$\mathbb{E}[(\gamma^{\oplus(f,s)})_\sigma] + \gamma(g,t) < \gamma(f,s) + \mathbb{E}[(\gamma^{\oplus(g,t)})_\sigma].$$

If $\hat{\tau}$ is our maximiser, we can then find a set $\Gamma \subseteq \mathsf{S}$ with $\mathbb{P}[((W_s)_{s\leq\hat{\tau}}, \hat{\tau}) \in \Gamma] = 1$, such that Γ is γ-monotone, that is,

$$\mathsf{SG} \cap \left(\Gamma^< \times \Gamma\right) = \varnothing,$$

where $\Gamma^< := \{(f,s) : \exists(\tilde{f},\tilde{s}) \in \Gamma, s < \tilde{s} \text{ and } f \equiv \tilde{f} \text{ on } [0,s]\}$. Denote the set of maximisers of P_γ by Opt_γ and consider another Borel function $\tilde{\gamma} : \mathsf{S} \to \mathbb{R}$. In Beiglböck et al. [2017b] it is shown that Opt_γ is non-empty and compact for suitable γ, and so we can assume that $\hat{\tau}$ is also a maximiser of the secondary optimisation problem

$$P_{\tilde{\gamma}|\gamma} := \sup\{\mathbb{E}[\tilde{\gamma}_\tau] : \tau \in \mathsf{Opt}_\gamma\}. \tag{1.4}$$

The set of secondary stop-go pairs, SG_2 consists of all $((f,s),(g,t))) \in \mathsf{S}\times\mathsf{S}$ such that $f(s) = g(t)$ and for every stopping time σ with $0 < \mathbb{E}[\sigma] < \infty$ we have

$$\mathbb{E}[(\gamma^{\oplus(f,s)})_\sigma] + \gamma(g,t) \leq \gamma(f,s) + \mathbb{E}[(\gamma^{\oplus(g,t)})_\sigma], \tag{1.5}$$

and the equality

$$\mathbb{E}[(\gamma^{\oplus(f,s)})_\sigma] + \gamma(g,t) = \gamma(f,s) + \mathbb{E}[(\gamma^{\oplus(g,t)})_\sigma] \tag{1.6}$$

implies the inequality

$$\mathbb{E}[(\tilde{\gamma}^{\oplus(f,s)})_\sigma] + \tilde{\gamma}(g,t) < \tilde{\gamma}(f,s) + \mathbb{E}[(\tilde{\gamma}^{\oplus(g,t)})_\sigma]. \tag{1.7}$$

Then we can also assume that

$$\mathsf{SG}_2 \cap \left(\Gamma^< \times \Gamma\right) = \varnothing.$$

Theorem 7.1 in Beiglböck et al. [2017b] tells us that there exists a γ-monotone Borel set $\Gamma \subseteq \mathsf{S}$ such that $\mathbb{P}$-a.s. $((W_t)_{t\le\hat{\tau}}, \hat{\tau}) \in \Gamma$.

We use the monotonicity principle in Chapter 3 to prove the existence of a new solution to the Skorokhod embedding problem known as the K-cave embedding.

1.3.3 The Root, Rost, and Cave Embeddings

Much of the work in this thesis concerns embeddings which can be seen as hitting times for (W_t, t), i.e. stopping times τ such that $\tau := \inf\{t \ge 0 : (W_t, t) \notin \mathcal{D}\}$ for some open region $\mathcal{D} \subseteq \mathbb{R} \times \mathbb{R}^+$. The first such example of these in the literature is the solution of Root [1969], and the Root stopping time is the hitting time of a barrier.

Definition 1.2. A barrier is a closed subset $\mathcal{B} \subseteq [-\infty, +\infty] \times [0, +\infty]$ such that

1. $(x, +\infty) \in \mathcal{B}$ for all $x \in [-\infty, +\infty]$
2. $(\pm\infty, t) \in \mathcal{B}$ for all $t \in [0, +\infty]$,
3. if $(x, t) \in \mathcal{B}$, then $(x, s) \in B$ for all $s \ge t$.

Loynes [1970] developed the ideas of Root by proving that a barrier $\mathcal{B}$ can always be written as

$$\mathcal{B} = \{(x, t) : t \ge R(x)\}$$

for some lower-semicontinuous barrier function $R : \mathbb{R} \to [0, \infty]$. Note that if $W_0 = 0$, then we can embed no mass beyond the points at which $R(x) = 0$. Loynes therefore defined *regular barriers* in the following way.

Definition 1.3. A barrier $\mathcal{B}$ generated by R is *regular* if R vanishes outside the interval $[x_-, x_+]$, where

$$\begin{aligned} x_- &:= \sup\{x < 0 : R(x) = 0\}, \\ x_+ &:= \inf\{x > 0 : R(x) = 0\}. \end{aligned}$$

By considering the minimum of two embedding barriers, Loynes proved that any barriers solving the embedding problem must have the same values of x_- and x_+, and must agree on $[x_-, x_+]$. In particular, he proves the following.

Theorem 1.4. *For any centred probability distribution with finite variance, there is exactly one regular barrier whose stopping time embeds the distribution and has finite expectation.*

Root barrier solutions were also considered in Rost [1976], where Rost proves that the Root stopping time solution to the embedding problem is the solution of minimal residual expectation, or equivalently, the Root stopping time minimises $\mathbb{E}\left[F(\tau)\right]$ over solutions of the Skorokhod embedding problem for any convex, increasing function $F(t)$ (equivalently, maximises for concave, decreasing F). This Root embedding is also considered in Cox and Wang [2013a] where the authors motivate this optimality property by using the arguments from Section 1.3.1 to show that the Root stopping time minimises the price of a variance call. The optimality is proved through the construction of a solution to the dual problem, i.e. a subhedging strategy. This construction will prove useful in our work on the highly-related cave and K-cave embeddings, and so we summarise the arguments here.

Suppose we wish to maximise $\mathbb{E}\left[F(\tau)\right]$ over solutions of (SEP) for a concave, decreasing function $F(t)$ with $F(0) = 0$ and right-derivative f. Suppose further that we have the Root solution to the embedding problem, $\tau_{\mathcal{D}}$, with continuation region $\mathcal{D}$ and stopping region $\mathcal{B}$ with barrier function R. Define

$$\begin{aligned}
M(x,t) &:= \mathbb{E}^{x,t}\left[f(\tau_{\mathcal{D}})\right], \\
Z(x) &:= \int_0^x \int_0^y M(z,0)\mathrm{d}z\mathrm{d}y, \\
G(x,t) &:= -\int_0^t M(x,s)\mathrm{d}s + Z(x), \\
H(x) &:= \int_0^{R(x)} \left(M(x,s) - f(s)\right)\mathrm{d}s - Z(x).
\end{aligned}$$

The authors prove that

- $G(x,t) + H(x) \geq F(t)$ for all (x,t),
- $G(W_{\tau_{\mathcal{D}}}, \tau_{\mathcal{D}}) + H(W_{\tau_{\mathcal{D}}}) \geq F(\tau_{\mathcal{D}})$,
- $G(W_t, t)$ is a supermartingale,
- $G(W_{t\wedge\tau_{\mathcal{D}}}, t\wedge\tau_{\mathcal{D}})$ is a martingale.

Then it follows that if τ is any other stopping time such that $W_\tau \sim \mu$, then

$$\begin{aligned}\mathbb{E}\left[F(\tau_{\mathcal{D}})\right] &= \mathbb{E}\left[G(W_{\tau_{\mathcal{D}}}, \tau_{\mathcal{D}}) + H(W_{\tau_{\mathcal{D}}})\right] \\ &= \mathbb{E}\left[G(W_0, 0) + H(W_{\tau_{\mathcal{D}}})\right] \\ &\geq \mathbb{E}\left[G(W_\tau, \tau) + H(W_\tau)\right] \\ &\geq \mathbb{E}\left[F(\tau)\right].\end{aligned}$$

It is also shown in Cox and Wang [2013a] that Root's barrier can be found as the solution to a particular variational inequality, and it is known that solutions to such variational inequalities are connected to the solutions of particular optimal stopping problems. This connection is exploited in Cox et al. [2018] to extend the Root solution of the embedding problem to the multi-marginal case. Similar approaches have been made in the case of the Rost embedding in McConnell [1991], Cox and Wang [2013b], Gassiat et al. [2015], and De Angelis [2015].

The Rost embedding stopping time, similar to the Root embedding, is a hitting time of a region known as a reversed or inverse barrier.

Definition 1.5. A reversed barrier is a closed subset $\mathcal{B}$ of $[-\infty, +\infty] \times [0, +\infty]$ such that

1. $(x, 0) \in \mathcal{B}$ for all $x \in [-\infty, +\infty]$,
2. $(\pm\infty, t) \in \mathcal{B}$ for all $t \in [0, +\infty]$,
3. if $(x, t) \in \mathcal{B}$, then $(x, s) \in \mathcal{B}$ for all $s \leq t$.

As in the Root embedding, for any reversed barrier $\mathcal{B}$ there is a unique, upper semi-continuous curve $R : \mathbb{R} \to [0, \infty]$ such that $\mathcal{B} = \{(x, t) :\ t \leq R(x)\}$, and Obłój [2004] observes that we can similarly define regular reversed barriers in which R must be increasing for $x > 0$ and decreasing for $x < 0$. Then an equivalent argument to Loynes [1970, Theorem 1] shows that there is at most one reversed barrier embedding any distribution. To argue that there is exactly one such reversed barrier we quote the following theorem based on Obłój [2004, Theorem 7.8].

Theorem 1.6. *For any probability measure μ on $\mathbb{R}$ with $\mu(\{0\}) = 0$, $\int x^2 \mu(\mathrm{d}x) = v < \infty$, and $\int x \mu(\mathrm{d}x) = 0$, there exists a reversed barrier $\mathcal{B}_\mu$ such that the stopping time*

$$\tau_{\mathcal{B}_\mu} := \inf\left\{t \geq 0 :\ (W_t, t) \in \mathcal{B}_\mu\right\}$$

solves (SEP)*, and* $\mathbb{E}\left[\tau_{\mathcal{B}_\mu}\right] = v$*. For any increasing, convex function* F *with* $F(0) = 0$*, this stopping time maximises* $\mathbb{E}\left[F(\tau)\right]$ *over solutions* τ *of the Skorokhod embedding problem for* μ*.*

This optimality is proved in Cox and Wang [2013b] for certain F using a superhedging approach, as in Cox and Wang [2013a].

Since the Root and Rost embeddings have optimality properties, they can be found using the monotonicity principle of Beiglböck et al. [2017b], and the authors also use the principle to prove the existence of a new embedding, the cave embedding, whose hitting region can be seen as the combination of a Root barrier and a Rost reversed barrier.

Definition 1.7. A region $\mathcal{B} \subseteq [-\infty, +\infty] \times [0, +\infty]$ is a cave barrier if there exists $t_0 \in (0, \infty)$, a reversed barrier $\mathcal{B}_0 \subseteq [-\infty, +\infty] \times [0, t_0]$, and a barrier $\mathcal{B}_1 \subseteq [-\infty, +\infty] \times [t_0, +\infty]$ such that $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$.

In Beiglböck et al. [2017b] it is proved that for a probability measure μ with $\int x^2 \mu(\mathrm{d}x) < \infty$ and $\mu(\{0\}) = 0$, there exists a cave barrier $\mathcal{B}$ such that $\tau_{\mathcal{B}} = \inf\{t \geq 0 : (W_t, t) \in \mathcal{B}\}$ minimises $\mathbb{E}[\varphi(\tau)]$ over all solutions of (SEP), where $\varphi : \mathbb{R}^+ \to [0, 1]$ is such that

- $\varphi(0) = 0$, $\lim_{t\to\infty} \varphi(t) = 0$, $\varphi(t_0) = 1$
- φ is strictly concave (and therefore increasing) on $[0, t_0]$
- φ is strictly convex (decreasing) on $[t_0, \infty)$.

We call functions of this form *cave-type* functions.

We will see in Chapter 3 that although the monotonicity principle provides a geometric structure for cave embeddings, this structure is not sufficient to uniquely determine the optimal cave barrier for the optimisation problem above. There are, in general, multiple cave barriers with stopping times which solve the embedding problem, but only one regular cave barrier which is optimal. We will show these results in the case of K-cave embeddings, to be introduced in Chapter 3, but all results are transferable to cave embeddings. In particular, these are the first examples of embeddings that are not uniquely determined by the geometric structure implied by the monotonicity principle.

1.3.4 Potential Theory

As stated earlier, Rost proves in Rost [1971] that for probability measures ν and μ, and a Markov process X with $X_0 \sim \nu$, there is a (possibly randomised) stopping time τ such that $X_\tau \sim \mu$ if and only if $\nu U^X \geq \mu U^X$. However, if $X = W$, a Brownian motion, then the potential kernel νU^W is infinite for positive measures ν, and it is therefore proposed in Obłój [2004] that we work instead with the one dimensional potential of a measure. In this section we give the definition of the potential and some results based on those from Chacon and Walsh [1976], Chacon [1977], and Obłój [2004] that will prove useful in later chapters. We also motivate the use of potential theory in embeddings by briefly introducing the solution of Chacon and Walsh [1976].

Definition 1.8. Denote by $\mathcal{M}^1$ the set of all probability measures on $\mathbb{R}$ with finite first moment, so $\mu \in \mathcal{M}^1$ iff $\int |x|\mu(\mathrm{d}x) < \infty$. Let $\mathcal{M}^1_m$ denote the subset of measures with expectation equal to m. The one-dimensional potential operator U acting from $\mathcal{M}^1$ into the space of continuous, non-positive functions $U : \mathcal{M}^1 \to C(\mathbb{R}, \mathbb{R}^-)$ is defined through $U_\mu(x) = -\int_{\mathbb{R}} |x - y|\mu(\mathrm{d}y)$, and we will refer to U_μ as the potential of μ.

Proposition 1.9. *Let $m \in \mathbb{R}$ and $\mu \in \mathcal{M}^1_m$. Then,*

1. *U_μ is concave, and Lipschitz continuous with parameter* 1
2. *$U_\mu(x) \leq U_{\delta_{\{m\}}}(x) = -|x - m|$ for all x*
3. *if $\nu \in \mathcal{M}^1$ and $U_\nu \leq U_\mu$ then $\nu \in \mathcal{M}^1_m$*
4. *for $\mu_1, \mu_2 \in \mathcal{M}^1_m$, $\lim_{|x|\to\infty} |U_{\mu_1} - U_{\mu_2}| = 0$*
5. *for $\mu_n \in \mathcal{M}^1_m$, $\mu_n \Rightarrow \mu$ iff $U_{\mu_n}(x) \to U_\mu(x)$ for all x as $n \to \infty$*
6. *for $\nu \in \mathcal{M}^1_m$, $U_\nu|_{[b,\infty)} = U_\mu|_{[b,\infty)}$ iff $\nu|_{[b,\infty)} = \mu|_{[b,\infty)}$*
7. *let $W_0 \sim \nu$ and define $\rho \sim W_{T_{a,b}}$, then $U_\nu|_{(-\infty,a]\cup[b,\infty)} = U_\rho|_{(-\infty,a]\cup[b,\infty)}$, and U_ρ is linear on $[a, b]$*
8. *if $(\mu_n)_n$ is a sequence of measures each with expectation m, then $\mu_n \to \mu$ as $n \to \infty$ if and only if $U_{\mu_n}(x) \to U_\mu(x)$ as $n \to \infty$ for all x*
9. *for any $x \in \mathbb{R}$, $\mu((-\infty, x]) = \frac{1}{2}(1 - (U_\mu)'(x+))$, and $\mu((-\infty, x)) = \frac{1}{2}(1 - (U_\mu)'(x-))$*
10. *if $W_0 \sim \nu$ and $W_\tau \sim \mu$ for a stopping time τ, then $\mathbb{E}[L^x_\tau] = U_\nu(x) - U_\mu(x)$.*

The Chacon-Walsh solution to the embedding problem uses potential theory to construct a sequence of distributions $(\mu_n)_n$, and corresponding embeddings $(\tau_n)_n$, such that the stopping times converge to some τ which embeds μ, the weak limit of the μ_n. Consider the simple case of a Brownian motion, W, started at 0, and suppose we wish to embed μ, where $U_\mu(x) \leq U_{\delta_0}(x)$ for all x. Let $\mu_0 = \delta_0$ and $\tau_0 = 0$. Take any x such that $U_\mu(x) < U_{\delta_0}(x)$ and consider the tangent of U_μ at x. This line intersects U_{δ_0} at two points $a_1 < x < b_1$. Let $\tau_1 := T_{a_1,b_1}$ be the first exit time of the Brownian motion from (a_1, b_1), and μ_1 the distribution of W_{τ_1}. At any point x, the value of the potential, $U_{\mu_1}(x)$, is the minimum of $U_{\mu_0}(x)$ and the tangent line evaluated at x. We repeat this procedure iteratively, choosing x such that $U_\mu(x) < U_{\mu_n}(x)$ and taking tangents to find a_n, b_n. Let $\tau_n = \tau_{n-1} + T_{a_n,b_n} \circ \theta_{\tau_n}$, where θ is the shift operator. Each U_{μ_n} is piecewise linear, and since any concave function can be written as the infimum of a countable number of affine functions, there exists a choice of the tangent lines such that $U_{\mu_n} \to U_\mu$. The stopping times τ_n are bounded and increase to a limit τ, and then $W_\tau \sim \mu$.

This is a simple description of an embedding using potential functions, however it also relates closely to the Azéma-Yor and Vallois solutions, see Cox [2008]. Dubins' solution from Dubins [1968] is a special case of the Chacon-Walsh construction.

Chapter 2

Discretisation of Optimal Skorokhod Embedding Problems

(This work has appeared in Cox and Kinsley [2017b])

In this chapter we prove a strong duality result for an infinite linear programming problem which has the interpretation of being a discretised version of (OptSEP), and we recover the continuous problem as a limit of the discrete problems. We show that primal optimisers, corresponding to optimal stopping rules for a random walk, exist and that if the objective function is chosen correctly then the optimal stopping time is the hitting time of a discrete version of a Root, Rost, or cave barrier. A limiting argument then allows us to reprove the existence of these embeddings.

The main strength of this approach is that we can derive properties of the discrete problem more easily than in continuous time, and then prove that these properties hold in the limit. For example, a consequence of the strong duality result is that dual optimisers exist, and our limiting arguments can be used to derive properties of the continuous time dual functions.

2.1 Introduction

Recall that the aim of this thesis is to analyse problems of the form

$$\sup_{\tau} \mathbb{E}\left[F\left(W_\tau, \tau\right)\right] \text{ over solutions to (SEP).} \tag{OptSEP}$$

In this chapter we discretise (OptSEP) to find an infinite linear programming problem which has a well-defined dual and where we are able to prove a strong duality result. For the optimal primal solution we can show that the discrete solution is also of the form of a hitting time, and then in the limit we can recover the continuous time embedding results of, for example, the Root and Rost solutions from Root [1969], Rost [1971, 1976], Chacon [1985]. We can also look at the limit of the dual optimisers and deduce properties of the continuous time dual functions. The dual optimisers in the case of the Root and Rost payoffs are found in Cox and Wang [2013a] and Cox and Wang [2013b] respectively, and the results of this chapter are used to find the same optimisers in Chapter 3.

In Beiglböck et al. [2017b], the authors use ideas from martingale optimal transport to show a monotonicity principle which can be used to to prove the existence of all known optimal solutions of (SEP). The monotonicity principle for an optimal Skorokhod embedding problem links to the idea of c-cyclical monotonicity in optimal transport theory, see Gangbo and McCann [1996]. The discrete counterpart introduced in this chapter is very reminiscent of the notion of *stop-go pairs* introduced in Beiglböck et al. [2017b], and using these ideas we prove the form of the stopping times in the discrete problems, as can be seen in Section 2.4.

As well as proving the existence of all known optimal solutions of (SEP), the monotonicity principle suggests that each solution has a geometric structure that is sufficient to uniquely determine the optimal stopping region. This was true for all previously known solutions, however the cave embedding introduced in Beiglböck et al. [2017b] is the first known example of a solution for which the geometric structure does not uniquely characterise the optimal stopping region, and we see in Chapter 3 that the K-cave embedding also has this problem of non-uniqueness. The results in this chapter allow us to find the form of the dual optimisers in the cave and K-cave embedding problems and therefore find a necessary and sufficient condition for the optimality of such barriers.

The use of discrete systems in model-independent finance is a common theme, see for example Acciaio et al. [2013], Beiglböck et al. [2013], Bayraktar et al. [2015], Bouchard et al. [2015], Badikov et al. [2017], and perhaps most similar to our approach, Neuberger [2007] and Hobson and Neuberger [2016a,b], however the exact nature of our problem, and the passage from discrete to continuous setups, appears to be novel.

This chapter is organised as follows: in Section 3.2 we set up the discrete linear programming problem by discretising (OptSEP). The dual problem is introduced in Section

3.3 and we give ideas on how this relates to the continuous time dual after proving a strong duality result. In Section 3.4 we show that the discrete optimal stopping region exhibits the same barrier-type properties as its continuous counterpart. Section 3.5 focuses on the convergence of the discrete problem back to (OptSEP). In Section 3.5.1 we show that we can discretise a feasible solution of (OptSEP) and then recover the same stopping time in the limit. Finally in Section 3.5.2 we prove that our discrete stopping region converges to a continuous stopping region with the same properties, and therefore that one limit of the discrete optimal solutions is an optimiser of (OptSEP).

2.2 Discretisation and Primal Formulation

We work on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_t)$ which supports a Brownian motion, W. Consider a measure μ with bounded support, so in particular $\exists x_*, x^*$ such that $x^* := \inf\{x > 0 : \ \mu((x,\infty)) = 0\}$ is the smallest $x > 0$ such that $\mu((x,\infty)) = 0$, and similarly $x_* := \sup\{x < 0 : \ \mu((-\infty, x)) = 0\}$. To embed this distribution into a Brownian motion, W, with a uniformly integrable stopping time, we know that we will stop immediately if we hit x^* or x_*, so we have absorbing barriers at these levels. We can split the interval $[x_*, x^*]$ into a uniform mesh $(x_j^N)_j$, for $j \in \{0, 1, \dots, L(N)\}$, where N is a parameter that we let go to infinity to reduce our mesh size and regain the continuous time case.

Now run W, stopping every time we hit a level x_j^N, and consider the process formed by this. In other words, consider the process $Y_k^N := W_{\tau_k^N}$, where $\tau_0^N := 0$ and if $W_{\tau_k^N} = x_j^N$ then $\tau_{k+1}^N := \inf\left\{t \geq \tau_k^N : W_t \in \left\{x_{j+1}^N, x_{j-1}^N\right\}\right\}$ for $k \geq 0$. We will later use Donsker's Theorem to recover our Brownian motion, and to ensure we can do this we need to choose the correct mesh size. We let $x_j^N = \frac{j}{\sqrt{N}}$ for $j \in \mathcal{J} := \{\lfloor x_*\sqrt{N}\rfloor, \lfloor x_*\sqrt{N}\rfloor + 1, \dots, \lfloor x^*\sqrt{N}\rfloor\}$. Let $j_0^N := \lfloor x_*\sqrt{N}\rfloor$, $j_1^N := \lfloor x_*\sqrt{N}\rfloor + 1$, $\dots$, $j_L^N := \lfloor x^*\sqrt{N}\rfloor$, where $L \sim \sqrt{N}$, so $\mathcal{J} = \{j_0^N, j_1^N, \dots, j_L^N\}$. We also define $\mathcal{J}' := \{j_1^N, \dots, j_{L-1}^N\}$, and $\mathcal{J}'' := \{j_2^N, \dots, j_{L-2}^N\}$.

If our Brownian motion has some stopping rule τ, then the discrete process also has some rule $\tilde{\sigma}$, defined to be the time t such that $\tau_{t-1}^N < \tau \leq \tau_t^N$. We consider the probabilities

$$
\begin{aligned}
p_{j,t}^N &= \mathbb{P}\left(Y_t^N = x_j^N,\ \tilde{\sigma} \geq t+1\right) \\
q_{j,t}^N &= \mathbb{P}\left(Y_t^N = x_j^N,\ \tilde{\sigma} = t\right).
\end{aligned}
$$

If our continuous process starts at $x^N_{j^*}$, for some j^*, then we have that $p_{j^*,0} = 1$, $p_{j,0} = 0$ for any $j \in \mathcal{J} \setminus \{j^*\}$, and $q_{j,0} = 0$ for all j. We also consider absorbing upper and lower barriers, so $p_{j^N_0,t}, p_{j^N_L,t} = 0$ for all t.

If there exists a maximiser τ of (OptSEP), we can discretise a Brownian motion with this stopping rule to create a random walk as above. This random walk will be a martingale and will embed some distribution μ^N, so the p, q associated to this stopping rule will be feasible solutions to the following problem:

$$
\begin{aligned}
\mathcal{P}' :\ & \sup_{p,q} \sum_{\substack{j\in\mathcal{J}\\ t\geq 1}} \bar{F}^N_{j,t} q_{j,t} \quad \text{over } (p_{j,t})_{\substack{j\in\mathcal{J}'\\ t\geq 1}}, (q_{j,t})_{\substack{j\in\mathcal{J}\\ t\geq 1}} \\
\text{subject to } & \bullet\ p_{j,t}, q_{j,t} \geq 0, \quad \forall j,t \\
& \bullet\ \sum_{t=1}^{\infty} q_{j,t} = \mu^N(\{x^N_j\}), && \forall j \in \mathcal{J} \\
& \bullet\ p_{j,t} + q_{j,t} = \frac{1}{2}(p_{j-1,t-1} + p_{j+1,t-1}), && \forall t \geq 2, j \in \mathcal{J}'' \\
& \bullet\ p_{j^N_1,t} + q_{j^N_1,t} = \frac{1}{2} p_{j^N_2,t-1}, && \forall t \geq 2 \\
& \bullet\ p_{j^N_{L-1},t} + q_{j^N_{L-1},t} = \frac{1}{2} p_{j^N_{L-2},t-1}, && \forall t \geq 2 \\
& \bullet\ p_{j^*+1,1} + q_{j^*+1,1} = \frac{1}{2}, \quad p_{j^*-1,1} + q_{j^*-1,1} = \frac{1}{2} \\
& \bullet\ p_{j,1} + q_{j,1} = 0, && \forall j \neq j^* \pm 1 \\
& \bullet\ q_{j^N_0,1} = 0 = q_{j^N_L,1} \\
& \bullet\ q_{j^N_L,t} = \frac{1}{2} p_{j^N_{L-1},t-1}, && \forall t \geq 2 \\
& \bullet\ q_{j^N_0,t} = \frac{1}{2} p_{j^N_1,t-1}, && \forall t \geq 2.
\end{aligned}
$$

If F is the function in (OptSEP) we are trying to maximise, then $\bar{F}$ will be a discrete version of F chosen so that $\bar{F}^N\left(\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor\right) \to F(x,t)$.

Note that for two different paths $\omega, \hat{\omega}$, we could have for example $\tau^N_3(\omega) < \tau^N_2(\hat{\omega})$, which means that in physical time, $q_{j,3}$ could be stopping mass before $q_{j,2}$. We will return to this limiting behaviour later, but currently we think instead of $\mathcal{P}'$ as describing the dynamics of a random walk on a fixed grid (x^N_j, t^N_n). The choice of our grid should be such that we can regain a Brownian motion in the limit as $N \to \infty$, and since $\mathbb{E}\left[\tau^N_k - \tau^N_{k-1} \middle| \mathcal{F}_{\tau^N_{k-1}}\right] = \frac{1}{N}$, we can quickly verify that $t^N_n = \frac{n}{N}$ is the correct time-step choice.

The aim of this discretisation is to allow us to appeal to primal-dual results in linear programming theory to learn something about the properties of the continuous time primal solution, and also our continuous time 'dual', the superhedging problem. The primal problem $\mathcal{P}'$ is an infinite problem, and so standard strong duality results do not apply. One option is to cut off our problem at some finite time T, use linear programming theory on this finite problem, and then recover our infinite time problem through letting $T \to \infty$. To avoid this extra limiting argument, we keep the infinite time scale, and instead work with a modified version of $\mathcal{P}'$ that will give an equivalent optimal value, but allows us to use results from infinite-dimensional programming.

These results rely largely on the existence of interior feasible points, and so our new problem must have inequality constraints. To allow this we drop the q variables from the formulation and instead just think of $q_{j,t} = \frac{1}{2}\left(p_{j-1,t-1} + p_{j+1,t-1}\right) - p_{j,t}$. We also change the embedding condition to a potential function constraint, requiring that the potential function of the terminal distribution of our process lies above that of μ^N. We denote the potential function of a measure μ by U_μ, so if μ is supported on $\mathbb{R}$, then $U_\mu(x) := -\int_{\mathbb{R}} |y-x|\mu(\mathrm{d}y)$ for $x \in \mathbb{R}$. Then for two measures μ and ν, the ordering of the potential functions $U_\mu(x) \geq U_\nu(x)$ for all x is equivalent to μ being less than ν in convex order.

For Brownian motion we know that $\mathbb{E}\left[L^x_\tau(W)\right] = \mathbb{E}\left[|W_\tau - x|\right] - \mathbb{E}\left[|W_0 - x|\right] = -U_\mu(x) + U_{\delta_0}(x)$ if $W_\tau \sim \mu$ and $(W_{t\wedge\tau})_t$ is uniformly integrable. In particular, if τ embeds a distribution with potential function greater than that of μ, then $\mathbb{E}\left[L^x_\tau(W)\right] \leq -U_\mu(x) + U_{\delta_0}(x)$. The discrete time analogue of the expected local time accrued at x^N_j is, in this case, $\sum_t p_{j,t}$, and so this condition becomes $\sum_{t=0}^\infty p_{j,t} \leq U^N_j$, where U^N_j is defined as

$$U^N_j := \sqrt{N}\left(\sum_i |x^N_i - x^N_j|\mu^N(\{x^N_i\}) - |x^N_{j^*} - x^N_j|\right).$$

Since we know that each U_j is finite, we have immediately that $(p_{j,t}) \in l^1$ when we consider it as an infinite sequence. For the conditions of strong duality we need to work with a smaller set than l^1, but we later show that our smaller space forms a set of stopping times which are dense in the set of stopping times from the l^1 problem, and we can recover the result in l^1. The idea is that we restrict ourselves to $(p_{j,t}) \in l^1(\boldsymbol{\pi}^{-1}) := \left\{(p_{j,t})_{j,t} : \sum_{j,t}\left|p_{j,t}\pi^{-1}_{j,t}\right| < \infty\right\}$, where $\pi_{j,t}$ are the values of p we get from running the random walk and stopping only when we hit the boundaries $j = j^N_0, j^N_L$. It will be useful to know more about these probabilities $\pi_{j,t}$ so that we know how the $p_{j,t}$ must decay.

Note that there will be many points at which we must have $\pi = 0$ since the random walk simply cannot visit them, for example $\pi_{j,t} = 0$ whenever $|j - j^*| > t$, and also we can only visit every other point at each x-level, depending on whether the time is odd or even. Let $\mathcal{A} := \{(j,t) : \ j \in \mathcal{J}', t \geq 1, \pi_{j,t} > 0\}$. Then $\mathcal{A}^C = \{(j,t) : \ |j - j^*| > t, \text{ or } j - j^* - t \text{ is odd}\}$. We will usually restrict ourselves to $\mathcal{A}$.

We can think of the random walk given by the probabilities $\pi_{j,t}$ as a random walk on $\mathcal{J}$ with absorbing states at $x^N_{j^N_0}$ and $x^N_{j^N_L}$, and so for the following result it will be useful to recall the following definitions for Markov chains, based on those in Ferrari and Rolla [2015].

Definition 2.1. Let S be a countable set, and consider a Markov chain X_n on $\mathsf{S} \cup \{\partial\}$, where ∂ is a cemetery state. For a probability measure ν, we define νT_n to be the conditional distribution of the Markov chain at time n started with law ν given that it is not absorbed in ∂ until time n.

- ν is a *quasi-stationary distribution* (q.s.d.) if $\nu = \nu T_1$, in which case $\nu = \nu T_n$ for all $n \geq 1$.
- The *Yaglom limit* of a probability measure ν is $\lim_{n\to\infty} \nu T_n$ if this limit exists and is a probability measure.

Lemma 2.2. *The sequence $(\pi_{j,t})_{(j,t)\in\mathcal{A}}$ of probabilities*

$$\pi_{j,t} := \mathbb{P}\left(Y^N_t = x^N_j,\ t < H_{x^N_{j_L}}(Y) \wedge H_{x^N_{j_0}}(Y)\right)$$

is in l^1, and in particular there is a vector $(m_j)_j$ and some constant $\rho \in (0,1)$ such that for each j,

$$\frac{\pi_{j,t}}{\rho^t} \to m_j \qquad \textit{as } t \to \infty.$$

Proof. We can of course argue that $(\pi_{j,t}) \in l^1$ since $\sum_{t\geq 0} \pi_{j,t} = U^\pi_j$ for U^π the appropriate potential function, but we give a proof which allows us to deduce something of the form of the $\pi_{j,t}$.

Consider running the random walk with the above absorbing region up until some time t, where we have some distribution of the remaining mass. Paths at time t leaving the centre-most point, call it $\hat{j}$, take longest to be absorbed at the barriers, but all mass leaving this point will be absorbed in a finite time almost surely. In particular, if we fix a small $\varepsilon > 0$, then there is some large M such that

$$\begin{aligned} &\mathbb{P}\Big(\text{path leaving } (\hat{\jmath}, t) \text{ will be absorbed by } t+M\Big) \geq \varepsilon, \\ \text{or} \quad &\mathbb{P}\Big(\text{path leaving } (\hat{\jmath}, t) \text{ hasn't been absorbed by } t+M\Big) \leq 1-\varepsilon. \end{aligned}$$

By our choice of $\hat{\jmath}$, for any $s \geq t+M$ we have $\sum_j \pi_{j,s} \leq (1-\varepsilon)\sum_j \pi_{j,t}$. If we take $t=0$, then we know that $\sum_j \pi_{j,0} = 1$ and $\sum_{j,0\leq r<M} \pi_{j,r} \leq (L+1)M$, so by the above reasoning,

$$\begin{aligned} \sum_{j,t} \pi_{j,t} &= \sum_{\substack{j \\ 0\leq r<M}} \pi_{j,r} + \sum_{\substack{j \\ M\leq r<2M}} \pi_{j,r} + \sum_{\substack{j \\ 2M\leq r<3M}} \pi_{j,r} + \cdots \\ &\leq (L+1)M + (L+1)M(1-\varepsilon) + (L+1)M(1-\varepsilon)^2 + \cdots \\ &= (L+1)M\sum_{n=0}^{\infty}(1-\varepsilon)^n \\ &= \frac{(L+1)M}{\varepsilon} < \infty. \end{aligned}$$

We now have that $(\pi_{j,t}) \in l^1$, and can see that the sequence appears to have some approximate exponential decay.

We have a discrete-time Markov chain on $\mathcal{J} = \mathcal{J}' \cup \{j_0^N, j_L^N\}$, where j_0^N, j_L^N are absorbing states and $\mathcal{J}'$ is an irreducible set of transient states. We can easily find the leading eigenvalue and corresponding eigenvector of our transition matrix restricted to $\mathcal{J}'$, and this gives us a quasi-stationary distribution for the process. From results on Yaglom limits of periodic Markov chains, see for example Ferrari and Rolla [2015, Theorem 9], the q.s.d. has a Yaglom limit. We also know, from standard results, that the survival probability of the process decays exponentially, like ρ^t for $0 < \rho < 1$ the leading eigenvalue. Combining these results shows that $\frac{\pi_{j,t}}{\rho^t} \to m_j$, for $m_j = \sqrt{\frac{2}{L}}\sin\left(\frac{j\pi}{L}\right) > 0$ and $0 < \rho = \cos\left(\frac{\pi}{L}\right) < 1$.

□

With this result in mind, and for technical reasons, we restrict our $(p_{j,t})$ to $l^1(\lambda) := \left\{(p_{j,t})_{j,t} : \sum_j |p_{j,t}|\,\lambda^t < \infty\right\}$, for some constant $\lambda > \rho^{-1} > 1$. Note here that we are defining $l^1(\lambda)$ in such a way that for a fixed t, $p_{j,t}$ is multiplied by λ^t for all j. Recall that we are optimising over probabilities $(p_{j,t})_{j,t}$ for $j \in \mathcal{J}'$ and $t \geq 1$ our discrete time steps, each of length $\frac{1}{N}$ in continuous time. We choose a $j^{*,N} \in \mathcal{J}$ to start our random walk at such that $x^N_{j^*} = \frac{\lfloor j^*, N \rfloor}{N} \to 0$ as $N \to \infty$. Our primal problem is then as follows:

$$\begin{aligned}\mathcal{P}^N(\lambda): \sup_p \Bigg\{ & \sum_{\substack{j \in \mathcal{J}'' \\ t \geq 2}} \bar{F}^N_{j,t} \left(\frac{1}{2} \left(p_{j-1,t-1} + p_{j+1,t-1} \right) - p_{j,t} \right) + \sum_{t \geq 2} \bar{F}^N_{j^N_L,t} \frac{1}{2} p_{j^N_{L-1},t-1} \\ & + \textstyle\sum_{t\geq 2} \bar{F}^N_{j^N_0,t} \frac{1}{2} p_{j^N_1,t-1} + \textstyle\sum_{t \geq 2} \bar{F}^N_{j^N_{L-1},t} \left(\frac{1}{2} p_{j^N_{L-2},t-1} - p_{j^N_{L-1},t} \right) \\ & + \textstyle\sum_{t \geq 2} \bar{F}^N_{j^N_1,t} \left(\frac{1}{2} p_{j^N_2,t-1} - p_{j^N_1,t} \right) + \bar{F}^N_{j^*+1,1} \left(\frac{1}{2} - p_{j^*+1,1} \right) \\ & + \bar{F}^N_{j^*-1,1} \left(\frac{1}{2} - p_{j^*-1,1} \right) \Bigg\},\end{aligned}$$

over $(p_{j,t})_{(j,t) \in \mathcal{A}}$ subject to

- $(p_{j,t}) \in l^1(\lambda)$
- $p_{j,t} \geq 0, \quad \forall j, t$
- $\mathbf{1}\{j = j^*\} + \sum_{t=1}^{\infty} p_{j,t} \leq U^N_j, \qquad \forall j \in \mathcal{J}'$
- $p_{j,t} \leq \frac{1}{2}(p_{j-1,t-1} + p_{j+1,t-1}), \qquad \forall t \geq 2, j \in \mathcal{J}''$
- $p_{j^N_1,t} \leq \frac{1}{2} p_{j^N_2,t-1}, \qquad \forall t \geq 2$
- $p_{j^N_{L-1},t} \leq \frac{1}{2} p_{j^N_{L-2},t-1}, \qquad \forall t \geq 2$
- $p_{j^*+1,1} \leq \frac{1}{2}, \quad p_{j^*-1,1} \leq \frac{1}{2}$
- $p_{j,1} = 0, \qquad \forall j \neq j^* \pm 1.$

We leave the conditions at $t = 1$ as equalities for clarity, but as with the $\pi_{j,t}$ there will be many points we do not visit at which we must have $p = q = 0$. As we will see in the next section, our duality result requires the primal feasible space to have interior points, and therefore we do not want to include $p_{j,t}$ as a variable for these points, so we optimise only over $p_{j,t}$ for $(j,t) \in \mathcal{A}$. We can then fix $p_{j,t} \equiv 0$ for $(j,t) \notin \mathcal{A}$ in our objective function, so we do not optimise over these. From here on we take the convention of optimising over $p_{j,t}$ for $(j,t) \in \mathcal{A}$ and setting $p_{j,t} = 0$ for $(j,t) \notin \mathcal{A}$. To avoid repetition we will not state this in every result.

Convention. We optimise only over the $p_{j,t}$ for $(j,t) \in \mathcal{A}$. Whenever we consider a point (j,t), it will be implicitly assumed that $(j,t) \in \mathcal{A}$.

Denote by $\mathcal{P}^N$ the above problem without the restriction of $(p_{j,t}) \in l^1(\lambda)$, i.e. we just require $p \in l^1$, so $\mathcal{P}^N := \mathcal{P}(1)^N$. Let $\mathrm{P}^N(\lambda)$, $\mathrm{P}^N = \mathrm{P}^N(1)$ be the optimal values of the problems $\mathcal{P}^N(\lambda)$, $\mathcal{P}^N$ respectively. We will show later that $\mathrm{P}^N = \mathrm{P}^N(\lambda)$. We will also introduce dual problems which we will denote by $\mathcal{D}^N(\lambda), \mathcal{D}^N$ with optimal values $\mathrm{D}^N(\lambda), \mathrm{D}^N$.

Remark 2.3. Note that our problem is of the form

$$\sup_p \Phi(p) \quad \text{over } (p) \in l^1(\lambda)$$

subject to

- $p_{j,t} \geq 0 \quad \forall (j,t)$
- $Ap \geq B$,

where Φ is linear, and A, B are given by

$$A := \begin{pmatrix} -\sum_{t=1}^{\infty} p_{j,t} \\ -p_{j^*\pm 1,1} \\ \frac{1}{2}(p_{j-1,t-1} + p_{j+1,t-1}) - p_{j,t} \\ \dfrac{1}{2}p_{2,t-1} - p_{1,t} \\ \vdots \end{pmatrix}, \qquad B := \begin{pmatrix} -U_j \\ -\dfrac{1}{2} \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{2.1}$$

2.3 Duality

2.3.1 Strong Fenchel Duality

With our choice of primal problem $\mathcal{P}$, we construct a dual problem and show a strong Fenchel duality result using the following theorem from Borwein and Zhu [2006, Theorem 4.4.3].

Theorem 2.4. *Let* X *and* Y *be Banach spaces, let* $f : \mathsf{X} \to \mathbb{R} \cup \{\infty\}$ *and* $g : \mathsf{Y} \to \mathbb{R} \cup \{\infty\}$ *be convex functions and let* $A : \mathsf{X} \to \mathsf{Y}$ *be a bounded linear map. Define the primal and* *dual values* $\mathsf{p}, \mathsf{d} \in [-\infty, \infty]$ *by the Fenchel problems*

$$\begin{aligned}\mathsf{p} &= \inf_{x \in \mathsf{X}} \{f(x) + g(Ax)\} \\ \mathsf{d} &= \sup_{y^* \in \mathsf{Y}^*} \{-f^*(A^* y^*) - g^*(-y^*)\}.\end{aligned}$$

Then $\mathsf{p} = \mathsf{d}$*, and the supremum in the dual problem is achieved if either of the following hold*

(i) $0 \in \operatorname{core}(\operatorname{dom}(g) - A\operatorname{dom}(f))$ *and* f, g *are lower semi-continuous*

(ii) $A\operatorname{dom}(f) \cap \operatorname{cont}(g) \neq \varnothing$.

We briefly explain the notation in the theorem, and then use it to prove duality. For a functional $f : \mathsf{X} \to \mathbb{R} \cup \{\infty\}$, its convex conjugate is the function $f^* : \mathsf{X}^* \to \mathbb{R} \cup \{\infty\}$ given by $f^*(x^*) = \sup_{x \in \mathsf{X}} \{\langle x^*, x\rangle - f(x)\}$. Recall also that $\operatorname{dom}(f) = \{x : f(x) < \infty\}$, $\operatorname{cont}(g) = \{y : g \text{ is continuous at } y\}$, and for $\mathsf{S} \subseteq \mathsf{Y}$, $s \in \operatorname{core}(\mathsf{S})$ if $\cup_{\alpha > 0} \alpha(\mathsf{S} - s) = \mathsf{Y}$.

As mentioned earlier, we consider $\mathsf{X} = l^1(\lambda)$ and we think of Y as being $\mathbb{R}^{L+1} \times l^1(\lambda)$, where the first $L+1$ variables correspond to the U_j conditions. Elements of $\mathsf{Y}^* = \mathbb{R}^{L+1} \times l^\infty(\lambda^{-1})$ will be written as $y^* = (\nu_j, \eta_{j,t})$. We take f, g to be

$$\begin{aligned} f(p) &:= \begin{cases} -\Phi(p) = \sum_{j,t} \left(-\bar{F}^N_{j,t}\right)\left(\frac{1}{2}(p_{j-1,t-1} + p_{j+1,t-1}) - p_{j,t}\right), & p \geq 0 \\ \infty, & \text{otherwise} \end{cases} \\ g(y) &:= \begin{cases} 0, & y \geq B \\ \infty, & \text{otherwise,} \end{cases} \end{aligned}$$

and Φ, A, B are as in (2.1). With these functions it is clear that the optimal value p is then $-\mathrm{P}^N(\lambda)$. Also, f, g are convex and lower semi-continuous, and A is a linear, bounded operator. The corresponding conjugates are

$$\begin{aligned} f^*(x^*) &= \begin{cases} \infty, & x^*_{j,t} > \bar{F}^N_{j,t} - \frac{1}{2}\left(\bar{F}^N_{j+1,t+1} + \bar{F}^N_{j-1,t+1}\right) \text{ for any } (j,t) \\ \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right), & \text{otherwise} \end{cases} \\ g^*(y^*) &= \begin{cases} \infty, & \nu_j > 0 \text{ or } \eta_{j,t} > 0 \text{ for any } j, (j,t) \\ \sum_{j=1}^{L-1} \nu^*_j(-U_j) - \frac{1}{2}(\eta_{j^*+1,1} + \eta_{j^*-1,1}), & \text{otherwise.} \end{cases} \end{aligned}$$

Before showing that these functions satisfy condition (i) of the theorem, we find the dual problem. The dual operator A^* is the functional $A^* : \mathsf{Y}^* \to \mathsf{X}^*$ satisfying $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle$ $\forall x \in \mathsf{X}$. Now,

$$\begin{aligned}
\langle y^*, Ax\rangle &= \sum_{\substack{1\le i\le L-1\\ t\ge 2}} \eta_{j_i^N,t}\left(\frac{1}{2}(p_{j_{i-1}^N,t-1}\mathbf{1}\{i\ge 2\} + p_{j_{i+1}^N,t-1}\mathbf{1}\{i\le L-2\}) - p_{j_i^N,t}\right)\\
&\qquad\qquad + \sum_{j\in\mathcal{J}'} \nu_j\left(-\sum_{t\ge 1} p_{j,t}\right) + \eta_{j^*+1,1}\left(-p_{j^*+1,1}\right) + \eta_{j^*-1,1}\left(-p_{j^*-1,1}\right)\\
&= \sum_{\substack{j\in\mathcal{J}'\\ t\ge 1}} p_{j,t}\left(\frac{1}{2}\left(\eta_{j+1,t+1} + \eta_{j-1,t+1}\right) - \eta_{j,t} - \nu_j\right),
\end{aligned}$$

and so, for $y^* \in \mathsf{Y}^*$,

$$A^* y^* = \left(\frac{1}{2}\left(\eta_{j+1,t+1} + \eta_{j-1,t+1}\right) - \eta_{j,t} - \nu_j\right)_{\substack{j\in\mathcal{J}'\\ t\ge 1}}.$$

From this, we see that our Fenchel dual is

$$\mathsf{d} = \sup\left\{ -\sum_{j\in\mathcal{J}'} \nu_j U_j - \frac{1}{2}\left(\eta_{j^*+1,1} + \eta_{j^*-1,1}\right) - \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right)\right\}$$

over $(\nu_j)_{j\in\mathcal{J}'}, (\eta_{j,t})_{\substack{j\in\mathcal{J}\\ t\ge 1}}$, where $(\nu,\eta) \in l^\infty(\lambda^{-1})$

subject to (2.2)

- $\eta_{j,t}, \nu_j \ge 0, \qquad \forall j,t$ (2.3)
- $\frac{1}{2}\left(\eta_{j+1,t+1} + \eta_{j-1,t+1}\right) - \eta_{j,t} - \nu_j \le \bar{F}^N_{j,t} - \frac{1}{2}\left(\bar{F}^N_{j+1,t+1} + \bar{F}^N_{j-1,t+1}\right), \quad \forall j,t.$ (2.4)

All that remains to show is that we satisfy one of the two remaining conditions of the theorem.

Theorem 2.5. *With A, f, g defined as above, (i) of Theorem 2.4 holds. In particular, there is no duality gap, and the optimiser is attained in the dual problem.*

Proof. A typical element of $\mathrm{dom}(g) - A\mathrm{dom}(f)$ looks like

$$w = \begin{pmatrix} y_j + \sum_{t\geq 1} p_{j,t} \\ y_{j^*\pm 1,1} + p_{j^*\pm 1,1} \\ y_{j,t} + p_{j,t} - \frac{1}{2}(p_{j-1,t-1} + p_{j+1,t-1}) \\ y_{j_1^N,t} + p_{j_1^N,t} - \frac{1}{2} p_{j_2^N,t-1} \\ \vdots \end{pmatrix} \quad \text{for} \quad \begin{array}{lll} (p_{j,t}),(y_j,y_{j,t}) \in & l^1(\lambda) & \\ p_{j,t} \geq 0 & & \forall j \in \mathcal{J}',\ t \geq 1 \\ y_j \geq -U_j & & \forall j \in \mathcal{J}' \\ y_{j^*\pm 1,1} \geq -\frac{1}{2} & & \\ y_{j,t} \geq 0 & & \forall j \in \mathcal{J}',\ t \geq 2. \end{array}$$

We are required to show that for any $z \in \mathbb{R}^{L+1} \times l^1(\lambda)$ $\exists \alpha > 0$ and a w of the above form such that $\alpha w = z$. Take $z = (z_j, z_{j,t}) \in \mathbb{R}^{L+1} \times l^1(\lambda)$, consider a constant, γ, and let

$$\rho_{j,t}^{k,s} = \gamma\, |z_{k,s}|\, \mathbf{1}\{s > t\} \frac{\pi_{j,t}}{\pi_{k,s}},$$

for $j,k \in \mathcal{J}$ and $s,t \geq 1$, so that

$$\rho_{j,t}^{k,s} - \frac{1}{2}\left(\rho_{j-1,t-1}^{k,s} + \rho_{j+1,t-1}^{k,s}\right) = \begin{cases} 0, & s \neq t \\ -\gamma\, |z_{k,t}|\, \frac{\pi_{j,t}}{\pi_{k,t}}, & s = t \end{cases} \quad \text{for } j \in \mathcal{J}',$$

and similarly for the other terms. Then $\rho_{j,t} = \sum_{k,s} \rho_{j,t}^{k,s} \geq 0$ defines a set of probabilities such that,

$$\rho_{j,t} - \frac{1}{2}(\rho_{j-1,t-1} + \rho_{j+1,t-1}) = -\gamma \sum_k |z_{k,t}| \frac{\pi_{j,t}}{\pi_{k,t}} \leq -\gamma\, |z_{j,t}|\,,$$

$$\sum_t \rho_{j,t} = \gamma \sum_{k,s,t} \mathbf{1}\{s > t\} |z_{k,s}| \frac{\pi_{j,t}}{\pi_{k,s}} = \gamma \sum_{k,s} |z_{k,s}| \frac{\pi_{j,0} + \ldots + \pi_{j,s-1}}{\pi_{k,s}} \in (0,\infty), \tag{2.5}$$

since $\pi \in l^1$. Also, for any $\varepsilon > 0$ there is a T such that for every j and $t \geq T$, $\left|\frac{\pi_{j,t}}{\rho^t} - m_j\right| < \varepsilon$. Then

$$\begin{aligned} \sum_{j,t} \rho_{j,t}\lambda^t &= \gamma \sum_{j,t,k,s} \mathbf{1}\{s > t\} |z_{k,s}| \lambda^t \frac{\pi_{j,t}}{\pi_{k,s}} \\ &= \gamma \sum_{\substack{k,s \leq T \\ j,t}} \mathbf{1}\{s > t\} |z_{k,s}| \lambda^t \frac{\pi_{j,t}}{\pi_{k,s}} + \gamma \sum_{j,t \leq T} \pi_{j,t}\lambda^t \sum_{k,s > T} \frac{|z_{k,s}|}{\pi_{k,s}} \\ &\qquad\qquad + \textstyle\sum_{k,s>T} \frac{|z_{k,s}|}{\pi_{k,s}} \sum_{j,T<t<s} \frac{\pi_{j,t}}{\rho^t} (\lambda\rho)^t \\ &< \infty. \end{aligned}$$

The first sum is a finite sum of finite terms, so is finite, and the second is finite since $(z_{j,t}) \in l^1(\lambda) \subset l^1(\boldsymbol{\pi})$. Let $\overline{m} = \max_j m_j$ and $\underline{m} = \min_j m_j$. Then for the final sum,

$$\begin{aligned}
\sum_{k,s>T} \frac{|z_{k,s}|}{\pi_{k,s}} \sum_{j,T<t<s} \frac{\pi_{j,t}}{\rho^t} (\lambda\rho)^t &\leq \sum_{k,s>T} \frac{|z_{k,s}|}{\pi_{k,s}} \sum_{j,T<t<s} (\lambda\rho)^t (m_j + \varepsilon) \\
&\leq (\overline{m} + \varepsilon) \sum_{k,s>T} \frac{|z_{k,s}|}{\pi_{k,s}} \frac{(\lambda\rho)^s - 1}{\lambda\rho - 1} \\
&\leq \frac{\overline{m} + \varepsilon}{\lambda\rho - 1} \sum_{k,s>T} \frac{|z_{k,s}|\lambda^s}{m_k - \varepsilon} - \frac{\overline{m} + \varepsilon}{\lambda\rho - 1} \sum_{k,s>T} \frac{|z_{k,s}|}{\pi_{k,s}} \\
&\leq \frac{\overline{m} + \varepsilon}{\underline{m} - \varepsilon} \frac{1}{\lambda\rho - 1} \sum_{k,s>T} |z_{k,s}|\lambda^s - \frac{\overline{m} + \varepsilon}{\lambda\rho - 1} \sum_{k,s>T} \frac{|z_{k,s}|}{\pi_{k,s}} \\
&< \infty,
\end{aligned}$$

again since $(z_{j,t}) \in l^1(\lambda) \subset l^1(\boldsymbol{\pi})$, and this also has a finite limit as $\varepsilon \to 0$.

In particular, we have $(\rho_{j,t}) \in l^1(\lambda)$. For each j, t we need $y_j \geq -U_j$, $y_{j,t} \geq 0$, $\gamma > 0$ and $\alpha > 0$ such that

$$\begin{aligned}
z_j &= \alpha \left(y_j + \sum_{t \geq 1} \rho_{j,t} \right) = \alpha \left(y_j + \gamma \sum_{k,s,t} \mathbf{1}\{s > t\} |z_{k,s}| \frac{\pi_{j,t}}{\pi_{k,s}} \right), \\
z_{j,t} &= \alpha \left(y_{j,t} + \rho_{j,t} - \frac{1}{2}(\rho_{j-1,t-1} + \rho_{j+1,t-1}) \right) = \alpha \left(y_{j,t} - \gamma\pi_{j,t} \sum_k \frac{|z_{k,t}|}{\pi_{k,t}} \right).
\end{aligned}$$

Setting $\alpha = \gamma^{-1}$, for any choice of γ such that

$$0 < \gamma \leq \min_j \left\{ \left| 2U_j \left(z_j - 2 \sum_t \pi_{j,t} \sum_{k,s} \frac{|z_{k,s}|}{\pi_{k,s}} \right)^{-1} \right| \right\},$$

we have that

$$y_j = \gamma z_j - \gamma \sum_{k,s,t} \mathbf{1}\{s > t\} |z_{k,s}| \frac{\pi_{j,t}}{\pi_{k,s}} > \frac{1}{2} \gamma z_j - \gamma \sum_t \pi_{j,t} \sum_{k,s} \frac{|z_{k,s}|}{\pi_{k,s}} \geq -U_j$$

for all $j \in \mathcal{J}'$, and

$$y_{j,t} = \gamma z_{j,t} + \gamma \pi_{j,t} \sum_k \frac{|z_{k,t}|}{\pi_{k,t}} \geq 0,$$

for all $j \in \mathcal{J}'$ and $t \geq 1$. Now we can see that the scaling factor was necessary to ensure that even if y_j is negative, we can scale it so that it is still larger than $-U_j$. Note here that division by zero isn't an issue since $\sum_t \pi_{j,t} \sum_{k,s} \frac{|z_{k,s}|}{\pi_{k,s}}$ is a strict upper bound of $\sum_t \rho_{j,t}$ by (2.5) and we can therefore choose a smaller bound if $z_j = 2\sum_t \pi_{j,t} \sum_{k,s} \frac{|z_{k,s}|}{\pi_{k,s}}$. To see that $(y_{j,t}) \in l^1(\lambda)$ note that $(z_{j,t}), (\rho_{j,t}) \in l^1(\lambda)$ and $y_{j,t} = \gamma z_{j,t} + \frac{1}{2}(\rho_{j-1,t-1} + \rho_{j+1,t-1}) - \rho_{j,t}$. $\square$

2.3.2 The Dual Problem

In this section we state clearly the dual problem and hint at its relation to the corresponding continuous time problem. In Section 3.5.2 we make these ideas rigorous by showing that the dual optimisers converge to their continuous counterparts in the case of the K-cave embedding.

Our Fenchel dual problem is

$$\mathcal{D}^N(\lambda): \qquad \text{minimise} \left\{ \sum_{j \in \mathcal{J}'} \nu_j U_j + \frac{1}{2}\left(\eta_{j^*+1,1} + \eta_{j^*-1,1}\right) + \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right) \right\}$$

over $(\nu_j)_{j \in \mathcal{J}'}, (\eta_{j,t})_{\substack{j \in \mathcal{J} \\ t \geq 1}}$ subject to

- $(\nu, \eta) \in l^\infty(\lambda^{-1})$ (2.6)
- $\eta_{j,t}, \nu_j \geq 0, \qquad \forall j, t$ (2.7)
- $\frac{1}{2}\left(\eta_{j+1,t+1} + \eta_{j-1,t+1}\right) - \eta_{j,t} - \nu_j \leq \bar{F}^N_{j,t} - \frac{1}{2}\left(\bar{F}^N_{j+1,t+1} + \bar{F}^N_{j-1,t+1}\right), \qquad \forall j, t.$ (2.8)

In Lagrangian duality we know that we have duality exactly when the complementary slackness conditions hold, and this is also true here. Fix N and take the optimal dual solution $y^* = (\nu^*, \eta^*)$ and any primal feasible p, so $g(Ap) = 0$. Then,

$$
\begin{aligned}
\mathsf{d} &= -\sum_{j\in\mathcal{J}'} \nu_j^* U_j - \frac{1}{2}\left(\eta^*_{j^*+1,1} + \eta^*_{j^*-1,1}\right) - \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right) \\
&\le -\sum_{j\in\mathcal{J}'} \nu_j^* U_j - \frac{1}{2}\left(\eta^*_{j^*+1,1} + \eta^*_{j^*-1,1}\right) - \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right) \\
&\qquad\qquad + \sum_{\substack{j\in\mathcal{J}'\\ t\ge 2}} \eta^*_{j,t}\left(\frac{1}{2}\left(p_{j+1,t-1} + p_{j-1,t-1}\right) - p_{j,t}\right) \\
&\le \sum_{\substack{j\in\mathcal{J}'\\ t\ge 1}} p_{j,t}\left(\frac{1}{2}\left(\eta^*_{j+1,t+1} + \eta^*_{j-1,t+1}\right) - \eta^*_{j,t} - \nu^*_j\right) - \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right) \\
&\le \sum_{\substack{j\in\mathcal{J}'\\ t\ge 1}} p_{j,t}\left(\bar{F}^N_{j,t} - \frac{1}{2}\left(\bar{F}^N_{j+1,t+1} + \bar{F}^N_{j-1,t+1}\right)\right) - \frac{1}{2}\left(\bar{F}^N_{j^*+1,1} + \bar{F}^N_{j^*-1,1}\right) \\
&= f(p) + g(Ap).
\end{aligned}
$$

We have equality in the above inequalities, and p is a primal optimiser, if and only if the following complementary slackness conditions hold:

$$p_{j,t} > 0 \implies \frac{1}{2}\left(\eta_{j-1,t+1} + \eta_{j+1,t+1}\right) - \eta_{j,t} - \nu_j = \bar{F}^N_{j,t} - \frac{1}{2}\left(\bar{F}^N_{j+1,t+1} + \bar{F}^N_{j-1,t+1}\right) \quad (2.9)$$

$$q_{j,t} > 0 \implies \eta_{j,t} = 0 \quad (2.10)$$

$$\nu_j > 0 \implies \sum_{t=1}^{\infty} p_{j,t} = U_j, \quad (2.11)$$

where as always we define $q_{j,t} = \frac{1}{2}\left(p_{j-1,t-1} + p_{j+1,t-1}\right) - p_{j,t}$.

In Cox and Wang [2013b] and Cox and Wang [2013a] (see Section 1.3.3) the authors seek functions $G(x,t)$ and $H(x)$ defined such that for a feasible primal stopping time σ,

- $G(x,t) + H(x) \ge F(x,t)$ everywhere
- $G(W_t, t)$ is a supermartingale
- $F(W_\sigma, \sigma) = G(W_\sigma, \sigma) + H(W_\sigma)$
- $G(X_{t\wedge\sigma}, t\wedge\sigma)$ is a martingale.

They show that if G and H exist then they give the optimal dual solution and there is no duality gap. The first two conditions above are dual feasibility conditions, and the final two ensure optimality, so we can think of them as complementary slackness conditions. We can then see how we might recover these functions from our dual problem.

We require functions G, H such that $G+H \geq F$, and $G(W_t, t)$ is a supermartingale, and these conditions correspond to (2.3) and (2.4) respectively. If we define $\tilde{\eta}_{j,t} = \eta_{j,t} + \bar{F}^N_{j,t}$ for a general η, and $\tilde{\eta}^*_{j,t} = \eta^*_{j,t} + \bar{F}^N_{j,t}$ for the optimal η^*, then (2.3), (2.4) become

$$\begin{aligned} &\tilde{\eta}_{j,t} \geq \bar{F}^N_{j,t} && \forall (j,t) \\ &\tilde{\eta}_{j,t} - \frac{1}{2}\left(\tilde{\eta}_{j+1,t+1} + \tilde{\eta}_{j-1,t-1}\right) + \nu_j \geq 0 && \forall (j,t), \end{aligned}$$

so $\tilde{\eta}$ has the feasibility properties we are looking for. Indeed, in Section 3.5.2 we show

$$\lim_{N\to\infty} \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor} = G(x,t) + H(x).$$

Our complementary slackness conditions (2.9) and (2.10) then correspond to $G(W_t, t)$ being a martingale in the continuation region and attaining equality in the stopping region. From (2.9) we also deduce that $\lim_{N\to\infty} N\nu^*_{\lfloor\sqrt{N}x\rfloor} = \frac{1}{2}H''(x)$.

2.3.3 Strong Duality of $\mathcal{P}^N, \mathcal{D}^N$

We now have strong duality in the λ problems, and attainment in the dual problem. In this section we show that we have primal attainment in $\mathcal{P}^N$, the l^1 problem, and that the optimal value in this case agrees with the $l^1(\lambda)$ problem, $\mathrm{P}^N = \mathrm{P}^N(\lambda)$.

Lemma 2.6. *Choose $\bar{F}^N$ so that $\mathrm{P}^N < \infty$. Then $\mathrm{P}^N = \mathrm{P}^N(\lambda)$ and the supremum P^N is attained by some sequence $p^* \in l^1$.*

Proof. To show primal attainment in $\mathcal{P}^N$, we show that the feasible region of $\mathcal{P}^N$ is a compact subset of l^1, and therefore supremums are attained. It is well known that a metric space is compact if it is totally bounded and complete. We argue that completeness of our feasible region follows from the formulation of $\mathcal{P}^N$, in particular we have no strict inequalities, so if the limit of a sequence of feasible solutions exists, then it will also be feasible. To show total boundedness we use the Kolmogorov-Riesz Compactness Theorem equivalent for l^p spaces from Hanche-Olsen and Holden [2010], first proved in Fréchet [1908].

Theorem. *A subset of l^r, where $1 \leq r < \infty$, is totally bounded if and only if,* *(i) it is pointwise bounded, and*

(ii) for every $\varepsilon > 0$ there is some n so that, for every x in the given subset,

$$\sum_{k>n} |x_k|^r < \varepsilon^r.$$

It is clear that our sequences are pointwise bounded, but also note that for any sequence $(p_{j,t})$ in our feasible region, we have that $p_{j,t} \leq \pi_{j,t}\ \forall (j,t)$, so in particular,

$$\sum_{j,t>n} p_{j,t} \leq \sum_{j,t>n} \pi_{j,t}.$$

By Lemma 2.2, $(\pi_{j,t}) \in l^1$, and therefore $\forall \varepsilon > 0\, \exists n$ such that $\sum_{j,t>n} \pi_{j,t} < \varepsilon$, and we are done.

Our strong duality result, Theorem 2.4, proves that $\mathrm{D}^N(\lambda) = \mathrm{P}^N(\lambda)$, but we also have weak duality in the original discretised problem, so $\mathrm{D}^N \geq \mathrm{P}^N$, where $\mathrm{D}^N(\lambda)$ is the value of our dual problem taking $(\nu, \eta) \in \mathbb{R}^{L+1} \times l^\infty(\lambda^{-1})$, and D^N is the optimal dual value with $(\nu, \eta) \in \mathbb{R}^{L+1} \times l^\infty$. Also,

$$\begin{aligned} l^1(\lambda) \subseteq l^1 &\implies \mathrm{P}^N \geq \mathrm{P}^N(\lambda) \\ l^\infty \subseteq l^\infty(\lambda^{-1}) &\implies \mathrm{D}^N \geq \mathrm{D}^N(\lambda). \end{aligned}$$

Let p^* be an optimiser of $\mathcal{P}^N$, whose existence we have just proven. Since $l^1(\lambda) \subseteq l^1$, we either have that $p^* \in l^1(\lambda)$, or we can 'cut-off' p^* at some finite time, and we get a feasible solution to $\mathcal{P}^N(\lambda)$. Define $p^T_{j,t} = p^*_{j,t}\mathbf{1}\{t < T\}$, then we have an approximating sequence, $(p^T)_T$, of feasible solutions of $\mathcal{P}^N(\lambda)$, with $\sum_{j,t} \bar{F}^N_{j,t} q^T_{j,t} \to \sum_{j,t} \bar{F}^N_{j,t} q^*_{j,t}$ provided $\sum_{j,t} \bar{F}^N_{j,t} q^*_{j,t} < \infty$. We must therefore have $\mathrm{P}^N = \mathrm{P}^N(\lambda)$. □

Remark 2.7. If (ν^*, η^*) are the optimisers of $\mathcal{D}^N(\lambda)$ then, similarly to above, we can consider $\hat{\nu}_j = \nu^*_j$ and $\hat{\eta}^T_{j,t} = \begin{cases} \eta^*_{j,t}, & t < T \\ \eta^*_{j,t} \wedge \bar{F}^N_{j,t}, & t \geq T \end{cases}$. Then $(\hat{\nu}, \hat{\eta}^T)$ are $\mathcal{D}^N$-feasible for large T and certain $\bar{F}^N$. An important example is where $\bar{F}^N$ is the discretisation of some European call option which is decreasing in time. In this case there is some T^* such that $\bar{F}^N_{j,t} = 0$ for all $t \geq T$, and then we find $\mathrm{D}^N = \mathrm{D}^N(\lambda)$ and $(\hat{\nu}, \hat{\eta}^T)$ attain D^N for any $T > T^*$. This is the case of the leveraged exchange traded fund payoff in Chapter 3.

2.4 The Cave Embedding Case

In the convergence arguments of the following chapters, stopping times of Brownian motions will be denoted by some variation of the symbol τ, and the stopping times of random walks by σ.

So far we have made no assumptions on our functions $F, \bar{F}^N$ except that they give well-defined optimisation problems, and that $\bar{F}^N \to F$ in some sense. We now show that for certain choices of F, these discrete optimisation problems have certain properties that have already been shown in the continuous time case. In particular, we focus on the ideas of the Root, Rost, and cave embeddings, first given in Root [1969], Rost [1971], and Beiglböck et al. [2017b] respectively. We concentrate on the cave embedding example, since it is a combination of a Root barrier and a Rost inverse-barrier, and therefore incorporates the arguments of the other two problems. Cave embeddings and their optimality properties were given in 1.3.3.

Fix $t_0 \in \mathbb{R}_+$ and consider our payoff to be the negative of a cave-type function (so that we are still maximising), i.e. $F(x,t) = f(t) := -\varphi(t)$, so $\bar{F}^N_{j,t} = \bar{f}^N(t) = -\varphi\left(\frac{t}{N}\right)$. We argue that our optimal $p^*_{j,t}$ define a discretised cave barrier stopping region for the random walk, and that this stopping region embeds exactly our distribution μ^N, for each N. If we were working with the primal problem $\mathcal{P}'$ then it would be clear that we embed μ^N, however it appears that this could fail with the conditions of $\mathcal{P}$, since our potential function could sit above that of μ^N, so we actually embed a different distribution. From now on we will be working solely with the primal optimisers p^*, and so for ease of presentation we will drop the * notation. Recall that we write $q_{j,t} := \dfrac{1}{2}(p_{j-1,t-1} + p_{j+1,t-1}) - p_{j,t}$.

Lemma 2.8. *For each N, if σ^N is the stopping time of a random walk Y^N given by the primal optimisers p, then $Y^N_{\sigma^N} \sim \mu^N$.*

Proof. If we have equality in our potential function condition, then by uniqueness, our stopping rule must embed μ^N. To show this, we change our payoff function in a way that doesn't affect our problem or previous arguments, but ensures that it is never optimal to have a strict inequality in our potential condition. We know that if $W_\sigma \sim \mu$, then $\mathbb{E}[\sigma] = \int x^2 \mu(\mathrm{d}x)$ is fixed, and so we can add a linear function of time to our payoff without affecting how the optimal solution is obtained. In particular, this means that in this case we can make our payoff increasing everywhere (at least away from zero), by considering $F(x,t) + Ct$ and therefore $\bar{f}^N(t) + C\frac{t}{N}$ for some large constant C. This will change our dual problem, and our interpretations of η^* and ν^* become $G(x,t)+H(x)-F(x,t)+Ct$ and $\frac{1}{2}H''(x)+C$, respectively, by considering $\hat{G}(x,t)=G(x,t)+Ct-Cx^2$ and $\hat{H}(x)=H(x)+Cx^2$.

Now if we release mass at some (j,r), since $\bar{f}^N(t)+C\frac{t}{N}$ is increasing, it will go on to score more than its current value, meaning that we always run our process for as long as possible in order to be optimal. If we have some x_k such that $\sum_{t=1}^{\infty} p_{k,t} < U_k$, then it is easy to see that we can find a site (i,s) such that $q_{i,s}>0$ and $U_i > \sum_t p_{i,t}$ and release some mass from here. Suppose there does not exist such a site (i,s). Then at any j at which we embed mass we have $U_j = \sum_t p_{j,t}$. Now, for any $j \neq j^* \pm 1$, $\sum_t p_{j,t} \le \sum_t \frac{1}{2}(p_{j-1,t-1}+p_{j+1,t+1}) = \frac{1}{2}\sum_t p_{j-1,t} + \frac{1}{2}\sum_{j,t} p_{j+1,t}$ with equality if and only if no mass is embedded at x_j^N. Then we have that $\sum_t p_{j,t}$ is linear between points j at which we embed mass, which we know is also case in potential functions, and therefore we must have $U_j = \sum_t p_{j,t}$ everywhere, by the convexity of U. Therefore, by contradiction, we have such a point (i,s).

Now fix $0<\varepsilon<\min\{q_{i,s}, U_i-\sum_t p_{i,t}\}$, and define $\bar{p},\bar{q}$ by

$$
\begin{aligned}
\bar{q}_{i,s} &= q_{i,s}-\varepsilon, & \bar{p}_{i,s} &= p_{i,s}+\varepsilon,\\
\bar{q}_{i+1,s+1} &= q_{i+1,s+1}+\frac{1}{2}\varepsilon, & \bar{q}_{i-1,s+1} &= q_{i-1,s+1}+\frac{1}{2}\varepsilon,\\
\bar{p}_{j,r} &= p_{j,r}, & \bar{q}_{j,r} &= q_{j,r} \quad \text{otherwise.}
\end{aligned}
$$

It is easy to check that $\bar{p},\bar{q}$ are $\mathcal{P}$-feasible, and also

$$
\begin{aligned}
\sum_{j,r}(\bar{f}^N(r)+C\frac{r}{N})\bar{q}_{j,r} &= \sum_{j,r}(\bar{f}^N(r)+C\frac{r}{N})q_{j,r} - \varepsilon(\bar{f}^N(s)+C\frac{s}{N})\\
&\qquad\qquad +\varepsilon(\bar{f}^N(s+1)+C\tfrac{s+1}{N})\\
&= \sum_{j,r}(\bar{f}^N(r)+C\frac{r}{N})q_{j,r} + C\frac{\varepsilon}{N} + \varepsilon(\bar{f}^N(s+1)-\bar{f}^N(s))\\
&\ge \sum_{j,r}(\bar{f}^N(r)+C\frac{r}{N})q_{j,r},
\end{aligned}
$$

if we choose $C \ge \bar{f}^N(1)-\bar{f}^N(2)$ for all N. Note that for each N there is a $C(N)<\infty$ such that $C(N) \ge \bar{f}^N(1)-\bar{f}^N(2)$, but also, if $\partial_t F(x,t)=\varphi'(t)$ is bounded, then $\lim_{N\to\infty} C(N) < \infty$ and we can choose $C=\sup_N C(N) = \lim_{N\to\infty} C(N)$. This shows we can always improve our payoff if we do not have equality in the potential condition, and therefore it is never optimal to have a strict inequality in this condition. In particular, by uniqueness of potentials, the optimal p embed μ^N into the associated random walk. $\square$

Remark 2.9. To ensure that our stopped process has the correct local time, and therefore embeds the correct distribution, we have, without loss of generality, made the function f increasing, or equivalently made $F(W_t, t)$ a submartingale. If $F(W_t, t)$ is a submartingale, then it is optimal to run our process for as long as possible before stopping. Since we want to ensure that we embed the correct distribution, we chose to bound our expected local time from above, using $\sum_t p_{j,t} \leq U_j$, or equivalently $\mathbb{E}\left[L^x_\tau(W)\right] \leq -U_\mu(x) + U_{\delta_0}(x)$. Then our process will only stop if continuing would violate this inequality for some x.

If $F(W_t, t)$ is a supermartingale, then it would be optimal in this setup to stop the process immediately. Note that in $\mathcal{P}^N$ we do not allow stopping at time 0, however the random walk can stop after one step, and in the limit this corresponds to the Brownian motion stopping immediately. To ensure we satisfy the embedding condition in this case we could instead choose to bound the expected local time from below, i.e. take $\mathbb{E}\left[L^x_\tau(W)\right] \geq -U_\mu(x) + U_{\delta_0}(x)$. Then the paths would stop as soon as the local time condition was satisfied.

Now that we know that we embed the correct distribution, we can actually show that we do this using an almost-deterministic stopping region that has the form of a cave barrier. By almost-deterministic we mean that we have some fixed (deterministic) stopping region made up of atoms on which $p_{j,t} = 0$, and some fixed continuation region in which $q_{j,t} = 0$, but the end point of each atom could have $p_{j,t}q_{j,t} > 0$. At these points we still have some randomisation in the stopping, so our stopping rule is not truly deterministic.

The proof of the following result is based on the idea of stop-go pairs, as introduced in Beiglböck et al. [2017b]. We consider some $j \in \mathcal{J}'$ and two paths: one which stops mass at x^N_j at some time t, and another which allows mass to continue from the point (x^N_j, s) for some other time s, and consider swapping these paths. By this we mean that we move the possible continuations of the path at (x^N_j, s) onto the stopped path at (x^N_j, t), thus the continued path is now stopped, and the previously stopped path now continues. We show that unless these paths satisfy an appropriate geometric condition, then this procedure will result in an improved payoff, and we therefore cannot have the optimal solution. This is exactly the idea behind the monotonicity principle of Beiglböck et al. [2017b]. In this sense, the following is a discrete version of the monotonicity principle result for the cave embedding, and sheds some light on the continuous counterpart of this argument in Beiglböck et al. [2017b].

Theorem 2.10. *The optimal solution of the primal problem $\mathcal{P}$, where $\bar{F}_{j,t} = \bar{f}(t) = -\varphi\left(\frac{t}{N}\right)$ for φ a cave-type function, is given by $p_{j,t}$ which give a stopping region for a random walk with the cave barrier-like property*

$$\begin{aligned} &\textit{if } q_{i,t} > 0 \;\; \textit{for some } (i,t) \textit{ where } t < t_0, \;\; \textit{then } p_{i,s} = 0 \, \forall s < t, \\ &\textit{if } q_{i,t} > 0 \;\; \textit{for some } (i,t) \textit{ where } t > t_0, \;\; \textit{then } p_{i,s} = 0 \, \forall s > t. \end{aligned}$$

Proof. First consider the inverse-barrier to the left of $t = t_0$. To show this, suppose we have a feasible solution with $q_{i,t} > 0$ and $p_{i,s} > 0$ for some i and $s < t < t_0$. We take some $0 < \varepsilon < \min\{\frac{1}{2}q_{i,t}, p_{i,s}\}$ and show that we can improve our objective function (increase the payoff), by transferring ε of the mass that currently leaves (i,s) onto (i,t). To move the paths we need to know how this ε of mass behaves, and the following quantities 'track' an ε mass of particles leaving (i,s):

$$\begin{aligned} \tilde{p}_{i,s} = \varepsilon, \quad \tilde{p}_{j,s} &= 0 \quad \forall j \neq i, \quad \tilde{q}_{j,s} = 0 \quad \forall j, \\ \tilde{p}_{j,r+1} &= p_{j,r+1} \times (\text{the } \varepsilon \text{ mass from } (j+1,r) + \text{ the } \varepsilon \text{ mass from } (j-1,r)) \\ &= p_{j,r+1}\left(\frac{p_{j+1,r}}{p_{j+1,r}+p_{j-1,r}}\frac{\tilde{p}_{j+1,r}}{p_{j+1,r}} + \frac{p_{j-1,r}}{p_{j+1,r}+p_{j-1,r}}\frac{\tilde{p}_{j-1,r}}{p_{j-1,r}}\right) \\ &= p_{j,r+1}\frac{\tilde{p}_{j+1,r}+\tilde{p}_{j-1,r}}{p_{j+1,r}+p_{j-1,r}} \qquad \forall j \neq j_0^N, j_L^N, \, \forall r \geq s, \\ \tilde{q}_{j,r+1} &= q_{j,r+1}\frac{\tilde{p}_{j+1,r}+\tilde{p}_{j-1,r}}{p_{j+1,r}+p_{j-1,r}} \qquad \forall j \neq j_0^N, j_L^N, \, \forall r \geq s, \\ \tilde{p}_{0,r+1} &= p_{0,r+1}\frac{\tilde{p}_{1,r}}{p_{1,r}}, \quad \tilde{q}_{0,r+1} = q_{0,r+1}\frac{\tilde{p}_{1,r}}{p_{1,r}} \qquad \forall r \geq s, \\ \tilde{p}_{L,r+1} &= p_{1,r+1}\frac{\tilde{p}_{L-1,r}}{p_{L-1,r}}, \quad \tilde{q}_{L,r+1} = q_{L,r+1}\frac{\tilde{p}_{L-1,r}}{p_{L-1,r}} \qquad \forall r \geq s. \end{aligned}$$

Using the $\tilde{p}, \tilde{q}$, and the procedure described here, we can write down the $\bar{p}, \bar{q}$ corresponding to the system after the transfer of the mass. Note that in the above, and it what follows, we define $\tilde{p}, \bar{p}$, and then $\tilde{q}, \bar{q}$ follow from defining $\tilde{q}_{j,t} = \dfrac{1}{2}(\tilde{p}_{j-1,t-1} + \tilde{p}_{j+1,t-1}) - \tilde{p}_{j,t}$, and similarly for $\bar{q}$, but we write them out for clarity. Before time s nothing changes, and so we have

$$\bar{p}_{j,r} = p_{j,r}, \quad \bar{q}_{j,r} = q_{j,r} \quad \forall (j,r) \in \{(j,r) : \, 1 \leq r < s\}.$$

At s we stop ε particles that previously left (i, s):

$$\bar{p}_{i,s} = p_{i,s} - \tilde{p}_{i,s} = p_{i,s} - \varepsilon, \qquad \bar{q}_{i,s} = q_{i,s} + \varepsilon,$$
$$\bar{p}_{j,s} = p_{j,s}, \qquad \bar{q}_{j,s} = q_{j,s} \quad \forall j \neq i.$$

Between times s and t we have lost these stopped paths, so

$$\bar{p}_{j,r} = p_{j,r} - \tilde{p}_{j,r}, \quad \bar{q}_{j,r} = q_{j,r} - \tilde{q}_{j,r} \quad \forall (j,r) \in \{(j,r) : \ s < r < t\}.$$

At time t we release an extra ε paths that were previously stopped at (i, t), giving

$$\bar{p}_{i,t} = p_{i,t} - \tilde{p}_{i,t} + \varepsilon, \qquad \bar{q}_{i,t} = q_{i,t} - \tilde{q}_{i,t} - \varepsilon,$$
$$\bar{p}_{j,t} = p_{j,t} - \tilde{p}_{j,t}, \qquad \bar{q}_{j,t} = q_{j,t} - \tilde{q}_{j,t} \quad \forall j \neq i.$$

For $r > t$ we have the ε paths from (i, t), but we have now lost ε from (i, s), so we have

$$\bar{p}_{j,r} = p_{j,r} - \tilde{p}_{j,r} + \tilde{p}_{j,r-(t-s)}, \quad \bar{q}_{j,r} = q_{j,r} - \tilde{q}_{j,r} + \tilde{q}_{j,r-(t-s)} \quad \forall (j,r) \in \{(j,r) : \ t < r\}.$$

These $\bar{p}, \bar{q}$ are $\mathcal{P}^N$-feasible by Lemma 2.11, and in Lemma 2.12 we show that these do indeed increase our payoff.

The lemmas then tell us that whenever we have a primal-feasible solution p where there exists i and $s < t$ such that $q_{i,t} > 0$ and $p_{i,s} > 0$, we can improve optimality by moving mass between these points. In particular, since we know by linear programming theory that an optimiser exists, our optimal p cannot have this property, so we have a discrete form of an inverse barrier: if $q_{i,t} > 0$ for some (i, t) where $t < t_0$, then $p_{i,s} = 0 \ \forall s < t$.

For the barrier to the right of $t = t_0$ we can repeat the same procedure. Suppose we have a feasible solution with $q_{i,t} > 0$ and $p_{i,s} > 0$ for some i and $t_0 < t < s$. We take some $0 < \varepsilon < \min\{q_{i,t}, \frac{1}{2}p_{i,s}\}$ and again transfer ε of the mass that currently leaves (i, s) onto (i, t). To do this we use the same $\tilde{p}$ as above, and define $\hat{p}, \hat{q}$ to be the values after the transfer by

$$
\begin{aligned}
&\hat{p}_{j,r} = p_{j,r}, && \hat{q}_{j,r} = q_{j,r}, && \forall j, r < t \\
&\hat{p}_{i,t} = p_{i,t} + \tilde{p}_{i,s} = p_{i,t} + \varepsilon, && \hat{q}_{i,t} = q_{i,t} - \varepsilon && \\
&\hat{p}_{j,t} = p_{j,t}, && \hat{q}_{j,t} = q_{j,t}, && \forall j \neq i \\
&\hat{p}_{j,r} = p_{j,r} + \tilde{p}_{j,r+s-t}, && \hat{q}_{j,r} = q_{j,r} + \tilde{q}_{j,r+s-t}, && \forall j, t < r < s \\
&\hat{p}_{i,s} = p_{i,s} + \tilde{p}_{i,2s-t} - \varepsilon, && \hat{q}_{i,s} = q_{i,s} + \tilde{q}_{i,2s-t} + \varepsilon, && \\
&\hat{p}_{j,s} = p_{j,s} + \tilde{p}_{j,2s-t}, && \hat{q}_{j,s} = q_{j,s} + \tilde{q}_{j,2s-t}, && \forall j \neq i \\
&\hat{p}_{j,r} = p_{j,r} + \tilde{p}_{j,r+s-t} - \tilde{p}_{j,r}, && \hat{q}_{j,r} = q_{j,r} + \tilde{q}_{j,r+s-t} - \tilde{q}_{j,r}, && \forall j, r > t.
\end{aligned}
$$

Again, Lemma 2.11 and Lemma 2.12 show that these are $\mathcal{P}^N$-feasible and that if we have such points $(i,t),(i,s)$ then we can improve our payoff by moving mass. Therefore, if $q_{i,t} > 0$ for some (i,t) where $t_0 < t$, then $p_{i,s} = 0 \ \forall t < s$. $\square$

Lemma 2.11. *The $\bar{p}, \hat{p}$ defined above are $\mathcal{P}^N$-feasible.*

Proof. First note that

$$
\begin{aligned}
\tilde{p}_{j,r+1} + \tilde{q}_{j,r+1} &= (p_{j,r+1} + q_{j,r+1}) \frac{\tilde{p}_{j+1,r} + \tilde{p}_{j-1,r}}{p_{j+1,r} + p_{j-1,r}} \\
&= \frac{1}{2}(p_{j+1,r} + p_{j-1,r}) \frac{\tilde{p}_{j+1,r} + \tilde{p}_{j-1,r}}{p_{j+1,r} + p_{j-1,r}} \\
&= \frac{1}{2}(\tilde{p}_{j+1,r} + \tilde{p}_{j-1,r}),
\end{aligned}
$$

and so the ε of mass evolves as expected. Also, since $\tilde{p}_{j,s} \geq 0 \ \forall j$, by induction we have that $\tilde{p}_{j,r}, \tilde{q}_{j,r} \geq 0 \ \forall (j,r)$. Then, from the above,

$$
\tilde{p}_{j,r+1} \leq \frac{1}{2}(\tilde{p}_{j+1,r} + \tilde{p}_{j-1,r}),
$$

and we also know that $\tilde{p}_{i,s} = \varepsilon < p_{i,s}$ and $\tilde{p}_{j,s} = 0 \leq p_{j,s}$ otherwise. Then, by induction,

$$
0 \leq \tilde{p}_{j,r} \leq p_{j,r} \quad \forall (j,r),
$$

and so

$$
0 \leq \tilde{q}_{j,r} \leq q_{j,r} \frac{p_{j+1,r} + p_{j-1,r}}{p_{j+1,r} + p_{j-1,r}} = q_{j,r} \quad \forall (j,r).
$$

It is then clear immediately that $\bar{p}_{j,r}, \bar{q}_{j,r} \geq 0 \ \forall (j,r) \neq (i,s)$. Note that $\tilde{q}_{i,s} \leq \frac{\varepsilon}{2}$ and therefore $\varepsilon < \frac{1}{2} q_{i,s} < \frac{2}{3} q_{i,s} \implies \tilde{q}_{i,s} + \varepsilon \leq \frac{3}{2}\varepsilon < q_{i,s}$, so we also have that $\bar{q}_{i,s} \geq 0$.

The new system also embeds the same amount of mass at every level, for example for $j \neq i$, we can check that our potential condition will not change:

$$\begin{aligned}\sum_r \bar{p}_{j,r} &= \sum_r p_{j,r} + \sum_{r>t} \tilde{p}_{j,r-(t-s)} - \sum_{r>s} \tilde{p}_{j,r} \\ &= \sum_r p_{j,r} + \sum_{r>s} \tilde{p}_{j,r} - \sum_{r>s} \tilde{p}_{j,r} \\ &= \sum_r p_{j,r}.\end{aligned}$$

The $\hat{p}$ are similar. $\square$

Lemma 2.12. *The new primal solution reduces our objective function:*

$$\begin{aligned}&\textit{if } q_{i,t} > 0 \textit{ and } p_{i,s} > 0 \textit{ for some } i \textit{ and } s < t < t_0 \textit{ then } \sum f(r)\bar{q}_{j,r} \geq \sum f(r) q_{j,r}, \\ &\textit{if } q_{i,t} > 0 \textit{ and } p_{i,s} > 0 \textit{ for some } i \textit{ and } t_0 < t < s \textit{ then } \sum f(r)\hat{q}_{j,r} \geq \sum f(r) q_{j,r}.\end{aligned}$$

Proof. Consider first the case of the inverse barrier. We have $\{r > s\} = \{r + t - s < t_0\} \cup \{r < t_0 < r + t - s\} \cup \{t_0 \leq r\}$ and in the first region,

$$\bar{f}^N(r + t - s) - \bar{f}^N(t) > \bar{f}^N(r) - \bar{f}^N(s) \quad \text{for } r + t - s < t_0,$$

by the convexity of $\bar{f}^N$ on $[0, t_0]$. For $r < t_0 < r + t - s$,

$$\begin{aligned}\bar{f}^N(r + t - s) - \bar{f}^N(t) &> \bar{f}^N(t_0) - \bar{f}^N(t) && (\bar{f}^N \text{ strictly increasing on } [t_0, \infty)) \\ &> \bar{f}^N(t_0) - \bar{f}^N(s + t_0 - r) && (\bar{f}^N \text{ strictly decreasing on } [0, t_0]) \\ &> \bar{f}^N(r) - \bar{f}^N(s) && (\bar{f}^N \text{ convex on } [0, t_0]).\end{aligned}$$

Finally, for $t_0 \leq r$,

$$\begin{aligned}\bar{f}^N(r + t - s) - \bar{f}^N(t) &> \bar{f}^N(r) - \bar{f}^N(t) && (\bar{f}^N \text{ increasing on } [t_0, \infty)) \\ &> \bar{f}^N(r) - \bar{f}^N(s) && (\bar{f}^N \text{ decreasing on } [0, t_0]).\end{aligned}$$

Then by the definition of $\bar{q}$ in each of our time regions, we have

$$\begin{aligned}\sum_{j,r}\bar{f}(t)\bar{q}_{j,r} &= \sum_{j,r}\bar{f}(r)q_{j,r} + \varepsilon(\bar{f}(s)-\bar{f}(t)) - \sum_{r>s,j}\bar{f}(r)\tilde{q}_{j,r} + \sum_{r>t,j}\bar{f}(r)\tilde{q}_{j,r-(t-s)} \\ &= \sum_{j,r}\bar{f}(r)q_{j,r} + \varepsilon(\bar{f}(s)-\bar{f}(t)) - \sum_{r>s,j}\bar{f}(r)\tilde{q}_{j,r} + \sum_{r>s,j}\bar{f}(r+t-s)\tilde{q}_{j,r} \\ &= \sum_{j,r}\bar{f}(r)q_{j,r} + \varepsilon(\bar{f}(s)-\bar{f}(t)) + \sum_{r>s,j}(\bar{f}(r+t-s)-\bar{f}(r))\tilde{q}_{j,r} \\ &= \sum_{j,r}\bar{f}(r)q_{j,r} + \sum_{r>s,j}(\bar{f}(s)-\bar{f}(t))\tilde{q}_{j,r} + \sum_{r>s,j}(\bar{f}(r+t-s)-\bar{f}(r))\tilde{q}_{j,r} \\ &= \sum_{j,r}\bar{f}(r)q_{j,r} + \sum_{r>s,j}(\bar{f}(r+t-s)-\bar{f}(t))\tilde{q}_{j,r} - \sum_{r>s,j}(\bar{f}(r)-\bar{f}(s))\tilde{q}_{j,r} \\ &\geq \sum_{j,r}\bar{f}(r)q_{j,r}.\end{aligned}$$

Briefly, here is how we get the above. For the first line we have just written out $\bar{p}, \bar{q}$ in full. We know that we almost surely embed all of the released mass in finite time and so $\sum_{j,r}\tilde{q}_{j,r} = \varepsilon$, and this gives us the fourth equality above. Finally, as we noted above, $\bar{f}^N(r+t-s) - \bar{f}^N(t) > \bar{f}^N(r) - \bar{f}^N(s)$ for all $r > s$.

We can also show that this argument doesn't change if we add a linear function of time to our payoff, as mentioned in Lemma 2.8. Considering now $\bar{f}(t) + C\frac{t}{N}$ we have, in the notation above, $C\sum_{r>s,j}\frac{r}{N}\tilde{q}_{j,r} - C\sum_{r>t,j}\frac{r}{N}\tilde{q}_{j,r-(t-s)} = C\varepsilon\frac{t-s}{N}$, so,

$$\begin{aligned}\sum_{j,r}(\bar{f}(r)+C\frac{r}{N})\bar{q}_{j,r} &= \sum_{j,r}(\bar{f}(r)+C\frac{r}{N})q_{j,r} + \varepsilon(\bar{f}(s)-\bar{f}(t)) + C\varepsilon\frac{s-t}{N} \\ &\qquad - \sum_{r>s,j}(\bar{f}(r)+C\frac{r}{N})\tilde{q}_{j,r} + \sum_{r>t,j}(\bar{f}(r)+C\frac{r}{N})\tilde{q}_{j,r-(t-s)} \\ &= \sum_{j,r}(\bar{f}(r)+C\frac{r}{N})q_{j,r} + \varepsilon(\bar{f}(s)-\bar{f}(t)) - \sum_{r>s,j}\bar{f}(r)\tilde{q}_{j,r} \\ &\qquad + \sum_{r>t,j}\bar{f}(r)\tilde{q}_{j,r-(t-s)} \\ &\geq \sum_{j,r}(\bar{f}(r)+C\frac{r}{N})q_{j,r}.\end{aligned}$$

For the case of the right-hand barrier,

$$\begin{aligned}
\sum_{j,r} \bar{f}(t)\hat{q}_{j,r} &= \sum_{j,r} \bar{f}(r)q_{j,r} + \varepsilon(\bar{f}(s) - \bar{f}(t)) - \sum_{r>s,j} \bar{f}(r)\tilde{q}_{j,r} + \sum_{r>t,j} \bar{f}(r)\tilde{q}_{j,r-(t-s)} \\
&= \sum_{j,r} \bar{f}(r)q_{j,r} + \varepsilon(\bar{f}(s) - \bar{f}(t)) - \sum_{r>s,j} \bar{f}(r)\tilde{q}_{j,r} + \sum_{r>s,j} \bar{f}(r+t-s)\tilde{q}_{j,r} \\
&= \sum_{j,r} \bar{f}(r)q_{j,r} + \varepsilon(\bar{f}(s) - \bar{f}(t)) + \sum_{r>s,j} (\bar{f}(r+t-s) - \bar{f}(r))\tilde{q}_{j,r} \\
&\geq \sum_{j,r} \bar{f}(r)q_{j,r},
\end{aligned}$$

since $\bar{f}(r)$ is increasing for $r > t_0$, and similarly when we add a linear term. □

Remark 2.13. For the corresponding results in the cases where $\bar{F}^N_{j,t} = f(\frac{t}{N})$ for f convex or concave, just consider the Rost or Root parts of the above respectively and the arguments hold exactly.

Remark 2.14. Similarly to Lemma 2.8, if $F(x,t) = f(t)$ and $f'(t)$ is bounded, then we can consider $f(t) - Ct$ for some large constant C and show that actually $p_{j,t}q_{j,t} = 0$ for all (j,t). This is not immediately obvious from Theorem 2.10 since we could have finitely many points at which $p_{j,t} > 0, q_{j,t} > 0$. Since these points will disappear when we take limits, this isn't an important detail and we omit the proof.

2.5 Convergence of the Discrete Problem

2.5.1 Recovering the Continuous Optimiser

Now we have a full picture for the discrete problem, we can show that the sequence of linear programming problems indeed converges to our continuous problem. We first show that if we discretise the optimal continuous time solution, then we recover it in the limit of the discrete problems produced.

Let τ be a solution of (OptSEP) and $\tilde{\sigma}^N$ the corresponding stopping time of the random walk Y^N_t as defined earlier to be the time k such that $\tau^N_{k-1} < \tau \leq \tau^N_k$.

Theorem 2.15. *For a function $F(x,t)$ continuous in both variables, and a suitable discretisation $\bar{F}^N(j,t)$ chosen so that $\bar{F}^N(\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor) \to F(x,t)$, we have*

$$\bar{F}^N\left(\sqrt{N}Y^N_{\tilde{\sigma}^N}, \tilde{\sigma}^N\right) \xrightarrow{\mathbb{P}} F(W_\tau, \tau) \quad \text{as } N \to \infty.$$

In particular, when F is bounded,

$$\mathbb{E}\left[\bar{F}^N\left(\sqrt{N}Y^N_{\tilde{\sigma}^N}, \tilde{\sigma}^N\right)\right] \to \mathbb{E}\left[F(W_\tau, \tau)\right].$$

To prove this we use the following lemma. The techniques used in the proof are very similar to methods which have been used in proofs of the central limit theorem, see for example Durrett [2010].

Lemma 2.16. *For the $\tilde{\sigma}^N$ defined above, we have*

$$\begin{aligned} Y^N_{\tilde{\sigma}^N} &\to W_\tau \quad \textit{almost surely, and} \\ \frac{\tilde{\sigma}^N}{N} &\to \tau \quad \textit{in probability, as } N \to \infty. \end{aligned}$$

Proof. Note that for any ω, $\left|W_\tau(\omega) - Y^N_{\tilde{\sigma}^N}(\omega)\right| = \left|W_\tau(\omega) - W_{\tau^N_{\tilde{\sigma}}}(\omega)\right| < \frac{2}{\sqrt{N}}$, and so $Y^N_{\tilde{\sigma}^N} \to W_\tau$ almost surely.

For the stopping time convergence, let $M(t) = \sup\{k : \tau^N_k \leq t\}$ for any t. We claim that for any $T > 0, \varepsilon > 0$,

$$\mathbb{P}\left(\sup_{s \leq T}\left|\frac{M(s)}{N} - s\right| > \varepsilon\right) \to 0, \text{ as } N \to \infty.$$

If this is true, then $\frac{M(\tau)}{N} \xrightarrow{\mathbb{P}} \tau$ as $N \to \infty$, but since $M(\tau) = \tilde{\sigma}^N - 1$ we have our result. All that remains is to prove the claim.

Fix $T, \varepsilon > 0$ and let $X_n = \frac{n}{N} - \tau^N_n$, so $(X_n)_n$ is a martingale. If n_0 is such that $M(T) \leq n_0$, then

$$\begin{aligned} \sup_{s \leq T}\left(s - \frac{M(s)}{N}\right) &\leq \sup_{n \leq n_0}\left\{\tau^N_n - \frac{n-1}{N}\right\} \leq \sup_{n \leq n_0}\left\{\frac{1}{N} - X_n\right\}, \\ \sup_{s \leq T}\left(\frac{M(s)}{N} - s\right) &\leq \sup_{n \leq n_0}\left\{\frac{n}{N} - \tau^N_n\right\} = \sup_{n \leq n_0} X_n. \end{aligned}$$

Therefore,

$$\sup_{s \leq T}\left|\frac{M(s)}{N} - s\right| \leq \frac{1}{N} + \sup_{n \leq n_0}|X_n|.$$

Note that $\tau^N_n = \sum_{k=1}^n \omega^N_k$, where the ω^N_k are iid random variables with distribution equivalent to the distribution of the exit time of the interval $[-\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}]$ of a Brownian motion started at 0. Then, using that $W_t^2 - t$ and $W_t^4 - 6tW_t^2 + 3t^2$ are martingales, we have $\mathbb{E}\left[\omega_k^N\right] = \frac{1}{N}$ and $Var(\omega_k^N) = \frac{2}{N^2} - \frac{1}{3N^2} - \frac{1}{N^2} = \frac{2}{3N^2}$. Choose $n_0 = 2TN^{\frac{3}{2}}$, then for large N, by Chebyshev's inequality,

$$\begin{aligned}
\mathbb{P}\left(M(T) > n_0\right) &= \mathbb{P}\left(\tau_{n_0+1}^N < T\right) \\
&\leq \mathbb{P}\left(\left|\tau_{n_0+1}^N - \frac{n_0+1}{N}\right| \geq \frac{n_0+1}{N} - T\right) \\
&\leq \frac{4(n_0+1)^2}{9N^2(n_0+1-NT)^2} \\
&< \frac{\varepsilon}{3}
\end{aligned}$$

for all sufficiently large N. Also, $X_{n_0} = \sum_{k=1}^{n_0} X_k - X_{k-1} = \sum_{k=1}^{n_0}(\frac{1}{N} - \omega_k^N)$, the sum of iid mean zero random variables with variance $\frac{2}{3N^2}$. Then, by Doob's martingale inequality,

$$\mathbb{P}\left(\sup_{n \leq n_0} |X_n| \geq \varepsilon\right) \leq \frac{4T}{3\varepsilon^2\sqrt{N}} < \frac{\varepsilon}{3}, \text{ for large } N.$$

Finally,

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leq T}\left|\frac{M(s)}{N} - s\right| > \varepsilon\right) &\leq \mathbb{P}\left(\frac{1}{N} + \sup_{n\leq n_0}|X_n| > \varepsilon\right) + \mathbb{P}\left(M(T) > n_0\right) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3}.
\end{aligned}$$

□

Now we know how $Y_{\tilde{\sigma}^N}^N$ and $\tilde{\sigma}^N$ converge, we prove our theorem.

Proof of Theorem 2.15. Since $\frac{\tilde{\sigma}^N}{N} \xrightarrow{\mathbb{P}} \tau$, and $Y_{\tilde{\sigma}^N}^N \to W_\tau$ almost surely, and F is continuous in both variables, $F\left(Y_{\tilde{\sigma}^N}^N, \frac{\tilde{\sigma}^N}{N}\right) \xrightarrow{\mathbb{P}} F(W_\tau, \tau)$ as $N \to \infty$. Now,

$$\begin{aligned}
\left|\bar{F}^N\left(\sqrt{N}Y_{\tilde{\sigma}^N}^N, \tilde{\sigma}^N\right) - F(W_\tau,\tau)\right| &\leq \left|\bar{F}^N\left(\sqrt{N}Y_{\tilde{\sigma}^N}^N, \tilde{\sigma}^N\right) - F\left(Y_{\tilde{\sigma}^N}^N, \frac{\tilde{\sigma}^N}{N}\right)\right| \\
&\quad + \left|F\left(Y_{\tilde{\sigma}^N}^N, \frac{\tilde{\sigma}^N}{N}\right) - F(W_\tau,\tau)\right|,
\end{aligned}$$

but by the convergence of $\bar{F}^N$, for any ω and any $\varepsilon > 0\, \exists N_{\varepsilon,\omega}$ such that $\forall N \geq N_{\varepsilon,\omega}$,

$$\left|\bar{F}^N\left(\sqrt{N}Y_{\tilde{\sigma}^N}^N, \tilde{\sigma}^N\right)(\omega) - F\left(Y_{\tilde{\sigma}^N}^N, \frac{\tilde{\sigma}^N}{N}\right)(\omega)\right| < \varepsilon,$$

and therefore

$$\bar{F}^N \left(\sqrt{N} Y^N_{\tilde{\sigma}^N}, \tilde{\sigma}^N\right) \xrightarrow{\mathbb{P}} F(W_\tau, \tau) \quad \text{as } N \to \infty.$$

□

Remark 2.17. This result tells us that if we discretise the optimal continuous time solution, to get some feasible $p^{\tau,N}_{j,t}$, and then take the limit, we recover our optimal value, so in particular,

$$\mathrm{P}^N \geq \sum_{j,t} \bar{F}^N_{j,t} q^{\tau,N}_{j,t} \implies \lim_{N\to\infty} \mathrm{P}^N \geq \lim_{N\to\infty} \sum_{j,t} \bar{F}^N_{j,t} q^{\tau,N}_{j,t} = \mathbb{E}\left[F(W_\tau, \tau)\right].$$

Remark 2.18. Note that as a corollary of Lemma 2.16 we have that $\mu^N = \mathcal{L}\left(Y^N_{\tilde{\sigma}^N}\right) \to \mathcal{L}\left(W_\tau\right) = \mu$.

2.5.2 Convergence of Barriers

We know that in the case where the reward function F corresponds to a cave, Root, or Rost reward, for each N, P^N is attained by some $p^{*,N}$ which give us a barrier-type property for the random walk. In this section we show that these discrete barriers converge to barriers in the continuous space, and furthermore that the corresponding stopped paths also converge. Similar results are proved for the Root and Rost barriers in Root [1969], Cox and Peskir [2015] respectively. In both papers the authors consider a Brownian motion hitting atomic barriers, and then show the convergence of stopping times as the atomic barriers converge to barriers embedding the full distribution. We use these ideas, but also require an additional step to move from the random walk to the Brownian motion, and for this we use Donsker's Theorem.

Consider again the cave embedding case, then we know that for each $j \in \mathcal{J}'$ there is a largest time $\bar{l}^N_j < t_0$ such that $p_{j,t} = 0\ \forall t \leq \bar{l}^N_j$, and similarly a smallest time $\bar{r}^N_j > t_0$ such that $p_{j,t} = 0\ \forall t \geq \bar{r}^N_j$. This defines $\bar{l}^N_j$ and $\bar{r}^N_j$ for $j \in \mathcal{J}'$, but we also want the stopping region to include the absorbing barriers at $x^N_{j^N_0}$ and $x^N_{j^N_L}$, so for $i = 0, L$ we choose $\bar{l}^N_{j^N_i} = \bar{r}^N_{j^N_i} = \lfloor N t_0 \rfloor$. Denote this stopping region by $\hat{\mathcal{B}}^N$. Note that for each j we either have $q_{j,\bar{l}^N_j} > 0$, or $q_{j,s} = 0\ \forall s < t_0$, and similarly for $\bar{r}^N_j$. To find the corresponding stopping region for the Brownian motion in continuous time, we shift from discrete to continuous time, so let

$$\mathcal{B}^N := \left\{(x,t) : (x, \lfloor Nt \rfloor) \in \hat{\mathcal{B}}^N\right\},$$

i.e. $(x,t) \in \mathcal{B}^N \iff (x, \lfloor Nt \rfloor) \in \hat{\mathcal{B}}^N$.

To make use of Donsker's Theorem we introduce, for each N, the process $\left(W_t^{(N)}\right)_{t\geq 0}$, the continuous, rescaled version of the simple symmetric random walk Y^N, given by

$$W_t^{(N)} := Y^N_{\lfloor Nt \rfloor} + (Nt - \lfloor Nt \rfloor)\left(Y^N_{\lfloor Nt \rfloor + 1} - Y^N_{\lfloor Nt \rfloor}\right).$$

Define the following stopping times:

$$\begin{aligned}
\sigma^{N,n} &:= \inf\{t \geq 0 : (W_t^{(N)}, t) \in \mathcal{B}^n\} \\
\sigma^N &:= \sigma^{N,N} \\
\hat{\sigma}^N &:= \inf\{k \geq 0 : (Y_k^N, k) \in \hat{\mathcal{B}}^N\} \\
\tau^N &:= \inf\{t \geq 0 : (W_t, t) \in \mathcal{B}^N\}.
\end{aligned}$$

Note that $Y^N_{\lfloor Nt \rfloor} = W_t^{(N)}$ when $Nt \in \mathbb{N}$, and so $\left(Y^N_{\hat{\sigma}^N}, \frac{\hat{\sigma}^N}{N}\right) = \left(W^{(N)}_{\sigma^N}, \sigma^N\right)$ almost surely.

We now show that the stopped random walks converge to a stopped Brownian motion with the correct distribution, and therefore that P^N converges to the continuous time optimal value.

Lemma 2.19. *The cave barriers $\mathcal{B}^N$ converge (possibly along a subsequence) to another cave barrier $\mathcal{B}^\infty$, and $(W_{\tau^N}, \tau^N) \xrightarrow{\mathbb{P}} (W_{\tau^\infty}, \tau^\infty)$ as $N \to \infty$, where τ^∞ is the Brownian hitting time of $\mathcal{B}^\infty$.*

Proof. Define measures ρ_N on $[x_*, x^*] \times [0, \infty]$ by $\rho_N(\cdot) := \mathbb{P}\left((W_{\tau^N}, \tau^N) \in \cdot\right)$. Since $\tau^N \leq H_{x_*} \wedge H_{x^*}$ for all N, and $H_{x_*} \wedge H_{x^*}$ is an integrable stopping time, we know that $\forall \varepsilon > 0\, \exists y_\varepsilon$ such that $\mathbb{P}\left(\tau^N > y_\varepsilon\right) < \varepsilon\ \forall N$, and therefore there is a compact set $A \subseteq [x_*, x^*] \times \mathbb{R}_+$ such that $\rho_N(A) < \varepsilon\ \forall N$. In particular, the sequence (ρ_N) is tight, and then by Prokhorov's theorem, there exists some ρ_∞ such that $\rho_N \xrightarrow{w} \rho_\infty$ (perhaps after restricting to a suitable subsequence). What remains to show is that $\rho_\infty(\cdot) = \mathbb{P}\left((W_{\tau^\infty}, \tau^\infty) \in \cdot\right)$, and we follow the ideas of Root [1969] and Cox and Peskir [2015].

In Root [1969], Root maps the closed half plane onto a closed, bounded rectangle and defines a norm on the space of closed subsets of the half plane by $d(R,S) = \sup_{x \in R} \inf_{y \in S} r(x,y)$, where r is the metric induced by taking the Euclidean metric on the rectangle. Under d, the space of closed subsets of our half plane is a separable, compact metric space and the space of all cave barriers is a closed subspace of the space, so is compact. We have a sequence of regions $\mathcal{B}^N$, and then by compactness they converge (possibly after taking a further subsequence) to some cave barrier $\mathcal{B}^\infty$ in this norm. Denote the hitting time of $\mathcal{B}^\infty$ by τ^∞.

Consider first our Root barrier, and let $\mathcal{B}^N$ now just denote the barrier part of the stopping region. By Root [1969] we then know that the hitting times of the $\mathcal{B}^N$ converge to the hitting time of $\mathcal{B}^\infty$ in probability as $N \to \infty$, and therefore $W_{\tau^N} \xrightarrow{\mathbb{P}} W_{\tau^\infty}$ also (for example by considering $\tau^N \wedge \tau^\infty$ and $\tau^N \vee \tau^\infty$).

Now we consider $\mathcal{B}^N$ to be just the Rost inverse-barrier section of our stopping region. In Cox and Peskir [2015] the authors define the Rost inverse-barrier by curves $b : (0,\infty) \to \mathbb{R}\cup\{+\infty\}$ and $c : (0,\infty) \to \mathbb{R}\cup\{-\infty\}$ so that the Rost stopping time is $\tau_{b,c} = \inf\{t > 0 : W_t \geq b(t) \text{ or } W_t \leq c(t)\}$. We can also define our atomic stopping region like this for each N, giving a sequence of curves b^N, and apply the same arguments. All we need to show is that $\forall \varepsilon > 0$, $\exists n$ such that $\forall N \geq n$, $b^\varepsilon \geq b^N \geq b_\varepsilon$, where $b^\varepsilon(t) := b(t+\varepsilon) + \varepsilon$ and $b_\varepsilon(t) := b(t-\varepsilon) - \varepsilon$ are defined in Cox and Peskir [2015]. We know that each $\mathcal{B}^N$ has absorbing boundaries, and this must hold in the limit, i.e. $\lim_{N\to\infty} b^N(t) = b^\infty(t) = x^* \ \forall t \geq t_0$ and similarly for c, where b^∞ is the curve associated to $\mathcal{B}^\infty$. This means we can just consider the part of the Rost inverse-barriers that are in the compact region $\{x \in [x_* - 1, x^*], t \leq t_0\}$, but then in this region we have that the metric r is bounded, and so for any $\delta > 0$, we can find an $\varepsilon > 0$ such that $d(\mathcal{B}^N, \mathcal{B}) < \delta \implies b^\varepsilon \geq b^N \geq b_\varepsilon$, with d the above Root norm.

Then, combining the results of Root [1969] and Cox and Peskir [2015] by considering the minimum of the two stopping times, we have that $(W_{\tau^N}, \tau^N) \xrightarrow{\mathbb{P}} (W_{\tau^\infty}, \tau^\infty)$ as $N \to \infty$. $\square$

Lemma 2.20. *With the stopping times defined previously,*

$$\left| \left(W^{(N)}_{\sigma^N}, \sigma^N \right) - \left(W_{\tau^N}, \tau^N \right) \right| \xrightarrow{d} 0 \quad \text{as } N \to \infty.$$

Proof. We again consider the barrier and inverse-barrier hitting times separately and use the ideas of Root [1969] and Cox and Peskir [2015].

For the Rost inverse-barrier alone we consider the Girsanov Theorem approach of Cox and Peskir [2015]. Fix $\delta > 0$ and some $T > 0$ and let $\varepsilon = \frac{\delta}{8x^*}$. Donsker's Theorem tells us that for any fixed T there is a Brownian motion W such that

$\mathbb{P}\left(\sup_{0\le t\le T}\left|W_t - W_t^{(N)}\right| > \varepsilon\right) \to 0$ as $N\to\infty$. In particular, if

$$A^{N,\varepsilon} := \left\{\sup_{0\le t\le T}\left|W_t - W_t^{(N)}\right| \le \varepsilon\right\},$$

then $\exists N_0$ such that $N \ge N_0 \implies \mathbb{P}\left(\left(A^{N,\varepsilon}\right)^C\right) < \frac{\delta}{4}$. Let $W_t^{\pm\varepsilon} = W_t \pm \varepsilon t$ and denote the associated hitting times of $\mathcal{B}^N$ by $\tau^{N,\pm\varepsilon}$. The Girsanov Theorem tells us that W^ε is a $\mathbb{Q}_\varepsilon$-Brownian motion, where $\frac{\mathrm{d}\mathbb{Q}_\varepsilon}{\mathrm{d}\mathbb{P}}\Big|_{\mathcal{F}_t} = \exp\left(\varepsilon W_t - \dfrac{1}{2}\varepsilon^2 t\right)$, and similarly for $W^{-\varepsilon}$. Also, on $A^{N,\varepsilon}$, we have $\left|\sigma^N - \tau^N\right| \le \left|\tau^{N,\varepsilon} - \tau^{N,-\varepsilon}\right|$. For any $t \le T$, $N \ge N_0$,

$$\begin{aligned}
\left|\mathbb{P}\left(\sigma^N \ge t\right) - \mathbb{P}\left(\tau^N \ge t\right)\right| &\le \left|\mathbb{P}\left(\sigma^N \ge t\middle|\left(A^{N,\varepsilon}\right)^C\right) - \mathbb{P}\left(\tau^N \ge t\middle|\left(A^{N,\varepsilon}\right)^C\right)\right|\mathbb{P}\left(\left(A^{N,\varepsilon}\right)^C\right)\\
&\quad + \left|\mathbb{P}\left(\sigma^N \ge t|A^{N,\varepsilon}\right) - \mathbb{P}\left(\tau^N \ge t|A^{N,\varepsilon}\right)\right|\mathbb{P}\left(A^{N,\varepsilon}\right)\\
&\le \left|\mathbb{P}\left(\sigma^N \ge t|A^{N,\varepsilon}\right) - \mathbb{P}\left(\tau^N \ge t|A^{N,\varepsilon}\right)\right| + 2\mathbb{P}\left(\left(A^{N,\varepsilon}\right)^C\right)\\
&\le \frac{\delta}{2} + \left|\mathbb{P}\left(\tau^{N,\varepsilon} \ge t\right) - \mathbb{P}\left(\tau^{N,-\varepsilon} \ge t\right)\right|\\
&= \frac{\delta}{2} + \left|\mathbb{E}^{\varepsilon}\left[\mathbf{1}\{\tau^N \ge t\}\right] - \mathbb{E}^{-\varepsilon}\left[\mathbf{1}\{\tau^N \ge t\}\right]\right|\\
&= \frac{\delta}{2} + \left|\mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{Q}_\varepsilon}{\mathrm{d}\mathbb{P}} - \frac{\mathrm{d}\mathbb{Q}_{-\varepsilon}}{\mathrm{d}\mathbb{P}}\right)\Big|_{\mathcal{F}_{\tau^N}}\mathbf{1}\{\tau^N \ge t\}\right]\right|\\
&= \frac{\delta}{2} + \left|\mathbb{E}\left[\mathrm{e}^{\varepsilon W_{\tau^N} - \frac{1}{2}\varepsilon^2\tau^N}\left(1 - \mathrm{e}^{-2\varepsilon W_{\tau^N}}\right)\mathbf{1}\{\tau^N \ge t\}\right]\right|\\
&\le \frac{\delta}{2} + \mathrm{e}^{\varepsilon x^*}\left(1 - \mathrm{e}^{-2\varepsilon x^*}\right)\\
&\le \frac{\delta}{2} + 2\varepsilon x^*\\
&< \delta.
\end{aligned}$$

It follows that $\left|\sigma^N - \tau^N\right| \xrightarrow{d} 0$ as $N\to\infty$, and therefore $\left|\sigma^N - \tau^N\right| \xrightarrow{\mathbb{P}} 0$ since the limit is a constant. Then, $\left|W_{\sigma^N}^{(N)} - W_{\tau^N}\right| \xrightarrow{\mathbb{P}} 0$ also as $N\to\infty$, and therefore for the Rost part of the stopping time we have the required convergence in probability. To see that $\left|W_{\sigma^N}^{(N)} - W_{\tau^N}\right| \xrightarrow{\mathbb{P}} 0$, note that $\left|W_{\sigma^N}^{(N)} - W_{\tau^N}\right| \le \left|W_{\sigma^N}^{(N)} - W_{\sigma^N}\right| + \left|W_{\sigma^N} - W_{\tau^N}\right|$ and use Donsker's Theorem on the first term and the convergence of the stopping times in the second term.

Now consider just the Root barrier and fix $\varepsilon > 0$. Since $\mathcal{B}^N \to \mathcal{B}^\infty$ by Lemma 2.19, we can repeat the proof of Root [1969, Lemma 2.4] to show that $\exists N_0, n_0$ such that $N \ge N_0, n \ge n_0 \implies \mathbb{P}\left(\left|\sigma^{N,n} - \sigma^N\right| > \varepsilon\right) < \varepsilon$ and therefore $\left|\sigma^N - \sigma^{N,n}\right| \xrightarrow{\mathbb{P}} 0$ as $N, n\to\infty$. Again by Donsker's Theorem, for any $\eta > 0$, $\exists N_0(\eta,\varepsilon)$ such that $N \ge N_0(\eta,\varepsilon) \implies \mathbb{P}\left(A^{N,\eta}\right) > 1 - \frac{\varepsilon}{6}$. Then choose $\eta > 0$ such that

$$\mathbb{P}\left(\sup_{\eta<t<\varepsilon} W_t > 2\eta \text{ and } \inf_{\eta<t<\varepsilon} W_t > -2\eta\right) > 1 - \frac{\varepsilon}{6},$$

which is possible since the probability tends to 1 as $\eta \to 0$. Therefore for $N \geq N_0(\eta, \varepsilon)$,

$$\mathbb{P}\left(\sup_{\eta<t<\varepsilon} W_t^{(N)} > \eta, \inf_{\eta<t<\varepsilon} W_t^{(N)} < -\eta\right) \geq \mathbb{P}\left(A^{N,\eta}\right)\mathbb{P}\left(\sup_{\eta<t<\varepsilon} W_t - \eta > \eta, \inf_{\eta<t<\varepsilon} W_t + \eta < -\eta\right)$$
$$> 1 - \frac{\varepsilon}{3}.$$

Let $\tau^\pi := H_{x_{j_0^N}}(W) \wedge H_{x_{j_L^N}}(W)$ and $\sigma^{N,\pi} := H_{x_{j_0^N}}(W^{(N)}) \wedge H_{x_{j_L^N}}(W^{(N)})$. Since the hitting region with atoms only at $j = j_0^N, j_L^N$ is an example of a Rost barrier, we know from the above that $\left(W^{(N)}_{\sigma^{N,\pi}}, \sigma^{N,\pi}\right) \xrightarrow{d} (W_{\tau^\pi}, \tau^\pi)$ as $N \to \infty$, and so since we are working on a bounded domain, $\mathbb{E}\left[(W^{(N)}_{\sigma^{N,\pi}})^2\right] \to \mathbb{E}\left[W^2_{\tau^\pi}\right]$. Then,

$$\mathbb{E}\left[\sigma^N\right] \leq \mathbb{E}\left[\sigma^{N,\pi}\right] = \mathbb{E}\left[(W^{(N)}_{\sigma^{N,\pi}})^2\right] \to \mathbb{E}\left[W^2_{\tau^\pi}\right],$$

as $N \to \infty$, and also each $\mathbb{E}\left[\sigma^{N,\pi}\right]$ is bounded. We can therefore find a uniform bound on $\mathbb{E}\left[\sigma^N\right]$, and in particular $\exists T$ such that, by the Markov Inequality,

$$\mathbb{P}\left(\sigma^N \geq T\right) \leq \frac{\mathbb{E}\left[\sigma^N\right]}{T} < 1 - \frac{\varepsilon}{3} \quad \forall N.$$

We can also find $n_0, N_0 \geq N_0(\eta, \varepsilon)$ such that $d\left(\hat{\mathcal{B}}^N, \hat{\mathcal{B}}^n\right) < \eta$ for any $N \geq N_0, n \geq n_0$, and then can follow exactly the argument of Root [1969, Lemma 2.4] to get that $\left|\sigma^N - \sigma^{N,n}\right| \xrightarrow{\mathbb{P}} 0$ as $N, n \to \infty$. We can use a similar argument to the above to then show that $\left|W^{(N)}_{\sigma^N} - W^{(N)}_{\sigma^{N,n}}\right| \xrightarrow{\mathbb{P}} 0$, and so $\left|\left(W^{(N)}_{\sigma^N}, \sigma^N\right) - \left(W^{(N)}_{\sigma^{N,n}}, \sigma^{N,n}\right)\right| \xrightarrow{\mathbb{P}} 0$ as $N, n \to \infty$. Donsker's Theorem also shows that for any n, $\left(W^{(N)}_{\sigma^{N,n}}, \sigma^{N,n}\right) \xrightarrow{d} (W_{\tau^n}, \tau^n)$ as $N \to \infty$, and we prove this in Lemma 2.21. Combining these results and Lemma 2.19 gives the necessary convergence. □

Lemma 2.21. *If τ^n and $\sigma^{N,n}$ are as defined above, then*

$$\left(W^{(N)}_{\sigma^{N,n}}, \sigma^{N,n}\right) \xrightarrow{d} (W_{\tau^n}, \tau^n) \quad \text{as } N \to \infty.$$

Proof. By the choice of our discretisation, we know by Donsker's Theorem that for any $T > 0$, $(W_t^{(N)}; t \leq T) \xrightarrow{d} (W_t; t \leq T)$ as $N \to \infty$, and then by the Portmanteau Theorem,

$$\lim_{N\to\infty} \mathbb{P}\left((W_t^{(N)})_{t\leq T} \in \mathcal{K}\right) = \mathbb{P}\left((W_t)_{t\leq T} \in \mathcal{K}\right) \quad \text{for all Borel } \mathcal{K} \text{ with } \mathbb{P}\left(X \in \partial\mathcal{K}\right) = 0,$$

where we consider $\mathcal{K}$ to be a subset of the set of continuous paths, $\mathcal{C}[0, T]$.

Fix n and consider $W^{(N)}_{\sigma^{N,n}}$ and W_{τ^n} for $N \geq n$. Take some point $(t, x_i^n) \in \mathcal{B}^n$ and fix a closed interval, $\bar{B}(\varepsilon) = \{x_i^n\} \times [t - \gamma, t + \gamma]$, of width $\gamma < n^{-1}$ around this point. We show that the set of continuous paths which hit $\mathcal{B}^n$ for the first time in $\bar{B}(\varepsilon)$ is a

Borel set. Note that since we are working with discrete barriers, there is a smallest $y > x_i^n$ such that $y \in \mathcal{B}^N$, and $(\{y\} \times [t-\gamma, t+\gamma]) \cap \mathcal{B}^n \neq \emptyset$, and also a largest $z < x_i^n$ satisfying the same property. Now consider the sets

$$\begin{aligned}
\mathcal{K}_q^{\varepsilon} &:= \{f \in \mathcal{C}[0,T] : \ f(s) < y - \varepsilon \ \forall s \in [t-\gamma, q] \cap \mathbb{Q}\}, \\
\mathcal{K}_q^{\varepsilon,\delta} &:= \{f \in \mathcal{C}[0,T] : \ f(q) < x_i^n + \delta\} \cap \mathcal{K}_q^{\varepsilon}, \\
\mathcal{K}^{\varepsilon,\delta} &:= \bigcup_{\substack{q \in [t-\gamma, t+\gamma] \\ q \in \mathbb{Q}}} \mathcal{K}_q^{\varepsilon,\delta}, \\
\mathcal{K}^{y} &:= \bigcap_{\substack{\delta > 0 \\ \delta \in \mathbb{Q}}} \bigcup_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} \mathcal{K}^{\varepsilon,\delta}.
\end{aligned}$$

Then, since $\mathcal{K}_q^{\varepsilon}, \mathcal{K}_q^{\varepsilon,\delta}$ are Borel, $\mathcal{K}^y$ is also. Similarly we can define the above when considering z instead of y, with the opposite inequalities, and we would find that $\mathcal{K}^z$ is Borel. Since our barrier is a closed region, $\mathcal{K}^1 := \{f \in \mathcal{C}[0,T] : \ f$ doesn't hit $\mathcal{B}^n$ before time $t - \gamma\}$ is open in $\mathcal{C}[0,T]$, and therefore $\mathcal{K} := \mathcal{K}^1 \cap (\mathcal{K}^y \cup \mathcal{K}^z)$ is a Borel set. But $\mathcal{K}$ is exactly the set of paths which hit $\mathcal{B}^n$ for the first time in $\bar{B}(\varepsilon)$.

Now, $\partial\mathcal{K}$ is the set of paths which start at 0 and either hit $\bar{B}(\varepsilon)$ at times $t \pm \gamma$, or hit $\bar{B}(\varepsilon)$ anywhere but also hit $\mathcal{B}^n$ elsewhere first without passing through any atoms of $\mathcal{B}^n$. The probability a Brownian path hits at $t \pm \gamma$ or one of the finite number of end points of the atoms is 0, but if it touches an atom of the boundary elsewhere then it will almost surely pass through the atom. Therefore, $\mathbb{P}(W \in \partial\mathcal{K}) = 0$, so the Portmanteau Theorem applies. By the definition of $\mathcal{K}$, and since γ, i were arbitrary,

$$\left(W^{(N)}_{\sigma^{N,n}}, \sigma^{N,n}\right) \xrightarrow{d} (W_{\tau^n}, \tau^n) \quad \text{as } N \to \infty, \text{ for any } n.$$

□

Theorem 2.22. *If* P^N *is our discrete optimal value, and* τ *is the optimal continuous time stopping time, then*

$$\mathrm{P}^N \to \mathbb{E}\left[F\left(W_\tau, \tau\right)\right], \quad \textit{as } N \to \infty.$$

Proof. We know from our choice of $\bar{F}$ that $\|\bar{F}^N(x, Nt) - F(x,t)\|_\infty \to 0$ as $N \to \infty$.

Then by Lemma 2.20 and the boundedness of F, we have

$$\begin{aligned}\mathbb{E}\left[\left|\bar{F}^N\left(W^{(N)}_{\sigma^N}, N\sigma^N\right) - F\left(W_{\tau^N}, \tau^N\right)\right|\right] &\leq \mathbb{E}\left[\|\bar{F}^N(x, Nt) - F(x,t)\|_\infty\right] \\ &\quad + \mathbb{E}\left[\left|F\left(W^{(N)}_{\sigma^N}, \sigma^N\right) - F\left(W_{\tau^N}, \tau^N\right)\right|\right] \\ &\to 0, \quad \text{as } N \to \infty.\end{aligned}$$

Also, Lemma 2.19 shows that $(W_{\tau^N}, \tau^N) \xrightarrow{\mathbb{P}} (W_{\tau^\infty}, \tau^\infty)$ as $N \to \infty$ and so, since F is bounded,

$$\mathbb{E}\left[F\left(W_{\tau^N}, \tau^N\right)\right] \to \mathbb{E}\left[F\left(W_{\tau^\infty}, \tau^\infty\right)\right] \quad \text{as } N \to \infty,$$

where τ^∞ is the hitting time of a cave barrier such that $W_{\tau^\infty} \sim \mu$, since $\mathcal{L}\left(W_{\tau^\infty}\right) = \lim_{N\to\infty} \mathcal{L}\left(W^{(N)}_{\sigma^N}\right) = \lim_{N\to\infty} \mu^N = \mu$. Then, by the optimality of τ, combining these results gives

$$\mathrm{P}^N = \mathbb{E}\left[\bar{F}^N\left(Y^N_{\hat{\sigma}^N}, \hat{\sigma}^N\right)\right] = \mathbb{E}\left[\bar{F}^N\left(W^{(N)}_{\sigma^N}, N\sigma^N\right)\right] \to \mathbb{E}\left[F\left(W_{\tau^\infty}, \tau^\infty\right)\right] \leq \mathbb{E}\left[F\left(W_\tau, \tau\right)\right].$$

Theorem 2.15 gives the other inequality, as mentioned in Remark 2.17. □

2.6 Conclusions

Lemma 2.19 and Theorem 2.22 recover the cave embedding result proved using the monotonicity principle in Beiglböck et al. [2017b], that is, there exists a cave barrier such that the hitting time of that barrier minimises $\mathbb{E}\left[\varphi(\sigma)\right]$ over stopping times σ such that $W_\sigma \sim \mu$. In addition, these results characterise these boundaries/stopping times as the limits of solutions to a discrete problem. The equivalent results hold in the cases of the Root and Rost embeddings, and we show the result for K-cave embeddings in Chapter 3.

As well as this approach being a novel way of reproving the existence of these embeddings, it can also be used to derive properties of the continuous time problem which are not easily deduced otherwise. For example, our principal motivation for this work was to establish the form of the optimal superhedging portfolio of a European call option on a leveraged exchange traded fund, and without the work here it is not clear that such an optimal portfolio exists. In Section 2.3.2 we give an indication of how our dual optimisers $(\eta^{*,N}, \nu^{*,N})$ converge to functions with which we can superhedge our payoff F, and this is formalised in Section 3.5.2. Optimal superhedging portfolios are given for Root and Rost-type payoffs in Cox and Wang [2013a] and Cox and Wang [2013b] respectively, and we can also use this approach to recover those functions.

The discrete setup of this problem is robust in the sense that we can change the problem somewhat and hope to still prove strong duality and derive properties of the associated continuous time problem. By changing the conditions at $t = 1$ in $\mathcal{P}^N$ we can consider the problem where our random walk starts according to some more general initial distribution. The strong duality and convergence results above will still hold provided we choose a starting measure (for the Brownian motion) μ_0 such that $\mu_0 \preceq_c \mu$, and choose the discretisations μ_0^N carefully.

Here we consider the case where the full distribution of the process at the terminal time is known, but this approach could give more insight into the problem where the potential of the terminal distribution is only given at finitely many points. As discussed in Section 1.3.1, the Skorokhod embedding problem arises as a consequence of the Breeden-Litzenberger formula of Breeden and Litzenberger [1978]. This result allows us to calculate the terminal distribution of a price process if we can observe the prices of European call options on this process at some fixed terminal time for all possible strikes. A more realistic assumption is that we can observe the call prices at finitely many strikes, and then we can only calculate the potential of the terminal distribution at these points.

A restriction of the work is that we consider only measures supported on bounded domains. We believe that the extension to unbounded domains is possible with some adjustments and additional assumptions. Our strong duality result relies on the existence of interior points of the primal feasible region, and our weighting function λ^t ensures that such points exist. The constant λ was chosen using the decay rate of the process which runs until it hits the boundaries x_* or x^*. In an infinite domain we would need a different decay function, but a clever choice of this function should allow the result to be proved. Since results such as Donsker's Theorem only provide convergence to a Brownian motion in distribution, we need the boundedness of the payoff function F on the unbounded domain to ensure that we have convergence of the primal values.

Chapter 3

The K-cave Embedding

(This work has appeared in Cox and Kinsley [2017a])

In this chapter we consider a particular form of (OptSEP) which[1] is motivated by the pricing of a call option on a leveraged exchange traded fund, LETF, in particular we look at finding an arbitrage-free upper bound on the price of such an option. We use the idea of Stop-Go pairs from Beiglböck et al. [2017b] to find a new solution to the embedding problem called the K-cave embedding.

Much like the cave embedding, there are multiple K-cave barriers that solve the embedding problem, and much of this chapter is dedicated to finding the optimal barriers using the dual superhedging problem. We propose a condition for optimality and prove the sufficiency of the condition using techniques similar to those in Cox and Wang [2013a,b]. For the necessity of the condition we discretise the problem and use results in Chapter 2 to find dual optimisers.

3.1 Introduction and Formulation

An exchange traded fund (ETF) is a security traded on a stock exchange that tracks an index or basket of assets. An ETF is an ownership stake in a pool of assets, so a number of investors can share in a large, diverse portfolio, spreading the transaction costs across all investors. A regular ETF matches the benchmark index's performance 1:1, whereas a leveraged ETF (LETF) will most commonly match it 2:1 or 3:1, usually by holding

[1] We are grateful to Pierre Henri-Labordère for suggesting this problem to us.

daily futures contracts. Daily compounding means that LETFs do not maintain their 2:1 relative performance over time, only over a single day, and even then transaction costs and fees need to be subtracted. For example, if we have a traditional index ETF and a 2:1 LETF both trading at \$100, and the index increases by 10% that day, then our ETF is at \$110 whilst our LETF is now worth \$120. Our LETF met its goal on this individual day, but then these prices are now fixed, since our funds are compounded daily. If the following day our index sees a decrease of 9%, then the ETF is at \$100.10, but our LETF value decreases by 18% to \$98.4. We can clearly see that over time we will not maintain our 2:1 ratio.

The first LETF was released in 2006, and by 2016 there were over 200 LETFs available, most commonly with 125%, 200%, or 300% ratios. At the time of writing, the value of assets in the global ETF market is over \$3 trillion, and some investors expect it to double in size by 2021. LETFs are typically written on very liquid ETFs, with vanilla options traded on both the ETF and the LETF. This means that our assumption of observing European call option prices on the underlying ETF is a reasonable one. LETFs have been studied mathematically in recent literature, for example Cheng and Madhavan [2009], Zhang [2010], Avellaneda and Zhang [2010], and Ahn et al. [2015]. In particular, Zhang [2010] considers options on an LETF in terms of options on the underlying ETF, giving a closed form solution when the volatility of $\log(S_t)$ is deterministic, and numerical results fitting various models when the volatility is random.

We work on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_t)$ which is rich enough to support a Brownian motion, W, and a uniformly distributed $\mathcal{F}_0$-random variable. We set interest rates and transaction costs to 0, and consider the dynamics of the LETF with leverage ratio $\beta > 1$ that is rebalanced continuously. Let S_t, L_t be the prices of the ETF and LETF respectively, and suppose S is some continuous martingale on $\mathbb{R}^+$. Then the distribution μ obtained from the Breeden-Litzenberger formula is supported on $\mathbb{R}^+$, and we assume it is such that $\int_{\mathbb{R}^+} |x|^\beta \mu(\mathrm{d}x) < \infty$. Then the price process of the LETF is given by

$$L_t = S_t^\beta \exp\left(-\frac{\beta(\beta-1)}{2} V_t\right),$$

where V_t is the accumulated quadratic variation of $\log S_t$ up to time t. It is easy to verify that L_t is a martingale when S_t is. To avoid dealing with the accumulated log quadratic variation, we time change by setting $\tau_t := \inf\{s \geq 0 : V_s = t\}$ and $X_t := S_{\tau_t}$. But then,

$$\mathrm{d}\langle X\rangle_t = \mathrm{d}\langle S\rangle_{\tau_t} = S_{\tau_t}^2 \mathrm{d}V_{\tau_t} = X_t^2 \mathrm{d}t$$

and therefore X_t is a geometric Brownian motion (GBM). The payoff function for a

European call option on the time-changed LETF with strike $k > 0$ is

$$F_L(x,t) := \left(x^\beta \exp\left(-\frac{\beta(\beta-1)}{2} t \right) - k \right)_+ .$$

Write $h_L(x,t) := x^\beta \exp\left(-\frac{\beta(\beta-1)}{2}t\right)$ so that $h_L(X_t,t)$ is a martingale since X_t is.

The problem of finding an upper bound on the price of such an option is then equivalent to solving the optimal Skorokhod embedding problem

$$\sup_\tau \mathbb{E}\left[\left(X_\tau^\beta \exp\left(-\frac{\beta(\beta-1)}{2}\tau \right) - k \right)_+ \right] \quad \text{over stopping times } \tau \text{ such that } X_\tau \sim \mu, \tag{GBMOptSEP}$$

where X is a GBM, and in fact an exponential martingale. We conjecture a hitting time solution with a stopping region of the form shown in Figure 3-1, bounded by curves $l_L(x)$ and $r_L(x)$ giving the boundary of an inverse-barrier and a barrier region respectively (defined below), such that $l_L(x) \leq K_L(x) \leq r_L(x)$ and l_L is increasing. The curve $K_L(x) := \frac{2}{\beta(\beta-1)} \ln(\frac{x^\beta}{k})$ is such that $h_L(x, K_L(x)) = k$, so we only 'score' a positive payoff if we are absorbed by l_L, i.e. to the left of K_L. The example in Figure 3-1 contains an infinite section, and the barriers could also have spikes, we assume no differentiability on the curves l_L, r_L.

We show that the function r_L is such that $\underline{\mathcal{R}} := \{(x,t) : t \geq r_L(x)\}$ is a (Root) barrier. Note in particular that the closedness of $\underline{\mathcal{R}}$ implies that r_L is lower semi-continuous. Similarly, l_L is such that $\overline{\mathcal{R}} := \{(x,t) : t \leq l_L(x)\}$ is an inverse barrier, or reverse barrier.

Definition 3.1. For an increasing function $K : \mathbb{R} \to \mathbb{R}$, a K-cave barrier $\mathcal{R}$, is a closed subset of $(-\infty, +\infty) \times [0, \infty)$ such that $\mathcal{R} = \underline{\mathcal{R}} \cup \overline{\mathcal{R}}$, where

- $\underline{\mathcal{R}} \subseteq \{(x,t) : t \geq K(x)\}$ is a barrier,
- $\overline{\mathcal{R}} \subseteq \{(x,t) : t \leq K(x)\}$ is an inverse barrier.

K-cave barriers have a similar form to cave barriers, and much of the analysis in this chapter also applies to the cave embedding. In particular, the results we derive in Section 3.5.2 can be deduced for the cave embedding in essentially the same manner as in this chapter.

We will actually consider pricing two options, the first of which is the problem described by (GBMOptSEP). The second problem is very similar and is notable due to its

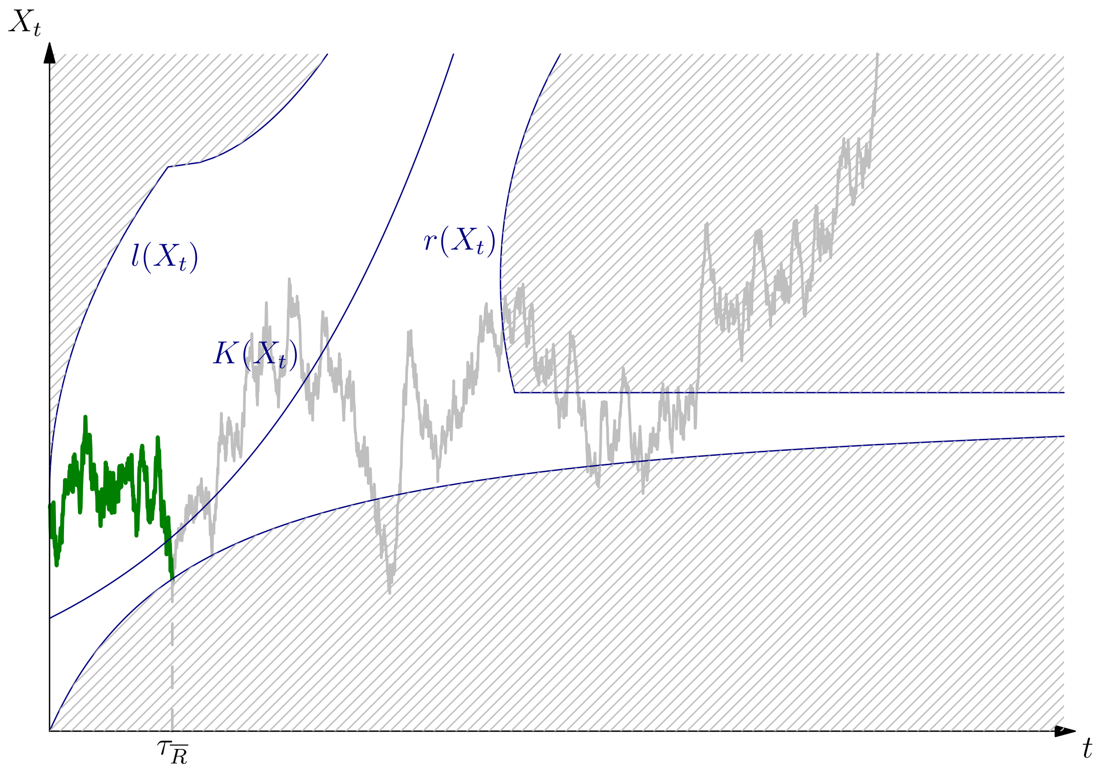

Figure 3-1: An example of our LETF problem that has a K-cave barrier with an infinite region

structure as a European call option on an exponential martingale. For this case, we consider the payoff function

$$F_{BM}(x,t) = \left(\exp\left(\beta x - \frac{1}{2}\beta^2 t\right) - k\right)_+$$

for $\beta > 0$ a constant, and $k > 0$ our strike. Here our underlying process is a standard Brownian motion, W, perhaps after a time change. We define h_{BM} to be $h_{BM}(x,t) := \exp(\beta x - \frac{1}{2}\beta^2 t)$ so that $h_{BM}(W_t, t)$ is a martingale, and we have a similar stopping region given by $l_{BM}(x), r_{BM}(x)$ seperated by $K_{BM}(x) = \frac{2x}{\beta} - \frac{2}{\beta^2}\ln(k)$, as shown in Figure 3-2. Our problem in this case is

$$\sup_\tau \mathbb{E}\left[\left(\exp\left(\beta W_\tau - \frac{1}{2}\beta^2\tau\right) - k\right)_+\right] \quad \text{over solutions of (SEP).} \qquad \text{(BMOptSEP)}$$

All results will be stated for a function $F \in \{F_L, F_{BM}\}$ and then proved for $F = F_{BM}$, with the simple adjustments to F_L explained. To make this clear, we will always be working in one of the following settings:

Assumption 1. *We write* $F(x,t) = F_{BM}(x,t) := \left(\exp\left(\beta x - \frac{1}{2}\beta^2 t\right) - k\right)_+$, *our underlying process is a Brownian motion* W_t *with* $W_0 = 0$ *and* μ *is a probability measure on* $\mathbb{R}$ *such that* $\int_{\mathbb{R}} \mathrm{e}^{\beta x}\mu(\mathrm{d}x) < \infty$.

Assumption 2. *We write* $F(x,t) = F_L(x,t) := \left(x^\beta \exp\left(-\frac{\beta(\beta-1)}{2}t\right) - k\right)_+$, *our underlying process is a geometric Brownian motion* X_t *with* $X_0 = 1$ *and* μ *is a probability measure on* $\mathbb{R}^+$ *such that* $\int_{\mathbb{R}^+} x^\beta \mu(\mathrm{d}x) < \infty$.

Moving between the two payoffs is generally simply since the problems are closely related by

$$X_t = X_0 \exp\left(W_t - \frac{1}{2}t\right) \implies X_t^\beta \exp\left(-\frac{\beta(\beta-1)}{2}t\right) = X_0^\beta \exp\left(\beta W_t - \frac{1}{2}\beta^2 t\right).$$

However, the embedding condition applies to different processes, and this is where the problems differ. In particular note that there are, in general, multiple stopping times τ which embed $W_\tau \sim \mu$, and so the distribution of W_τ is not enough to determine the distribution of $X_\tau = X_0 \exp\left(W_\tau - \frac{1}{2}\tau\right)$ due to the dependence on τ.

3.2 Existence of a Maximiser

3.2.1 Existence of K-cave Embeddings

We use the notion of stop-go pairs (see Section 1.3.2) to prove the following theorem:

Theorem 3.2. *Under the conditions of Assumption* 1, *there exists a stopping time* $\tau_{\mathcal{R}}$ *which maximises* $\mathbb{E}[F(W_\tau, \tau)]$ *over all solutions to* (SEP) *and which is of the form* $\tau_{\mathcal{R}} := \inf\{t > 0 : (W_t, t) \in \mathcal{R}\}$ *for some* K*-cave barrier* $\mathcal{R}$.

To prove this we consider the set of stop-go pairs of our primary and secondary, yet to be determined, optimality problems, and for these we need to introduce local times. The local time of a continuous semimartingale Z at a is the increasing, continuous process L^a that gives the Itô-Tanaka formula:

$$(Z_t - a)_+ = (Z_0 - a)_+ + \int_0^t \mathbf{1}_{\{Z_s > a\}} \mathrm{d}Z_s + \frac{1}{2}L_t^a.$$

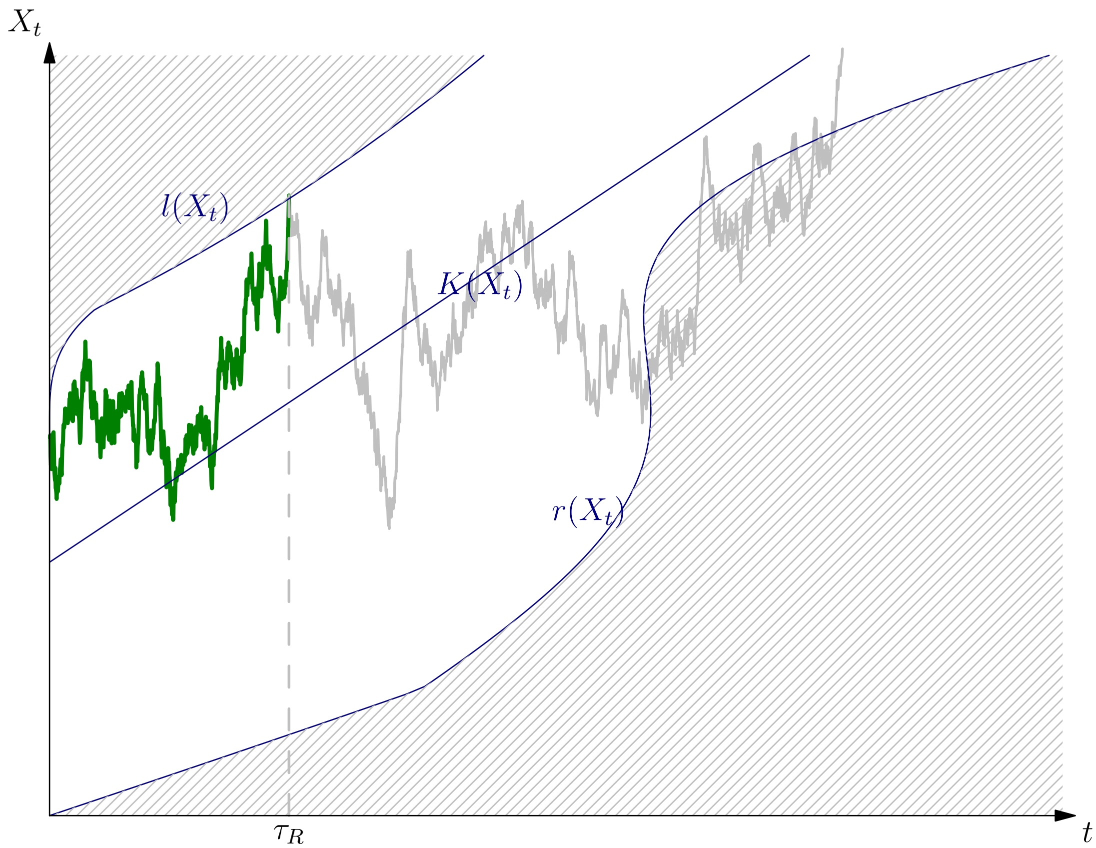

Figure 3-2: An example K-cave stopping region with continuous boundaries

Observe that we can write

$$L_t^a(Z) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t \mathbf{1}_{[a,a+\epsilon]}(Z_s) \mathrm{d}\langle Z \rangle_s,$$

where $\langle Z \rangle_s$ is the quadratic variation process of Z. By establishing the form of SG and SG_2 we can argue as in Beiglböck et al. [2017b] that we have a γ-monotone set supporting a maximiser of our two optimisation problems, and that it can be written as a stopping time of the required form.

Proof. We prove for $F = F_{BM}$. Write

$$M_u^f := h(f(s) + W_u, s + u) = \exp(\beta(f(s) + W_u) - \frac{\beta^2}{2}(s + u)) = h^f M_u$$
$$M_u^g := h(g(t) + W_u, t + u) = \exp(\beta(g(t) + W_u) - \frac{\beta^2}{2}(t + u)) = h^g M_u$$

where the constant $h^f = h(f(s), s)$ is introduced to emphasise that the process is of the form $(h(f(s),s)M_u)_u$, for $M_u = \exp(\beta W_u - \frac{\beta^2}{2}u)$ with $(W_u)_u$ a BM. We then see that, for $\alpha = \frac{h^g}{h^f} = \exp\left(-\frac{1}{2}\beta^2(t-s)\right)$,

$$M_u^g = \alpha M_u^f.$$

Then, after applying the Monotone Convergence Theorem along the localising sequence $\sigma_j = \sigma \wedge j$, along with Fatou's Lemma and Conditional Jensen's Inequality, the first term in (1.5) becomes

$$\mathbb{E}[F((f \oplus W)_{s+\sigma}, s+\sigma)] = (h^f - k)_+ + \frac{1}{2}\mathbb{E}[L_\sigma^k(M^f)].$$

Here we are taking the local time of the process $(M_u^f)_u$ accumulated at k up to time σ.

If we use the Itô-Tanaka formula on both sides of (1.5), this is equivalent to

$$(h^f - k)_+ + \frac{1}{2}\mathbb{E}[L_\sigma^k(M^f)] + (h^g - k)_+ \leq (h^g - k)_+ + \frac{1}{2}\mathbb{E}[L_\sigma^k(M^g)] + (h^f - k)_+,$$

which holds iff

$$\mathbb{E}[L_\sigma^k(M^f)] \leq \mathbb{E}[L_\sigma^k(M^g)].$$

We have equality, i.e. case (1.6), when

$$\mathbb{E}[L_\sigma^k(M^f)] = \mathbb{E}[L_\sigma^k(M^g)],$$

which clearly holds when $h^f = h^g$, which happens exactly when $s = t$, since $f(s) = g(t)$.

We aim to show that

$$h^f < h^g \leq k \implies \begin{cases} \text{either} & \mathbb{E}[L_\sigma^k(M^f)] < \mathbb{E}[L_\sigma^k(M^g)] \\ \text{or} & \mathbb{E}[L_\sigma^k(M^f)] = \mathbb{E}[L_\sigma^k(M^g)] = 0 \end{cases} \tag{3.1}$$

$$h^f > h^g \geq k \implies \begin{cases} \text{either} & \mathbb{E}[L_\sigma^k(M^f)] < \mathbb{E}[L_\sigma^k(M^g)] \\ \text{or} & \mathbb{E}[L_\sigma^k(M^f)] = \mathbb{E}[L_\sigma^k(M^g)] = 0. \end{cases} \tag{3.2}$$

We have to argue the two cases seperately, so suppose first that $h^f < h^g \leq k$ and take a stopping time σ such that $\mathbb{P}[M_\sigma^g > k] > 0$. Also let $\alpha = \frac{h^g}{h^f} = \exp(-\frac{\beta^2}{2}(t-s))$, so

$\alpha > 1$. Then we have

$$\begin{aligned}\mathbb{E}[L^k_\sigma(M^g)] &= 2\mathbb{E}[(M^g_\sigma - k)_+] - 2(h^g - k)_+\\ &> 2\mathbb{E}[(M^g_\sigma - \alpha k)_+] - 2(h^g - \alpha k)_+\\ &= \mathbb{E}[L^{\alpha k}_\sigma(M^g)],\end{aligned}$$

since $(h^g - k)_+ = (h^g - \alpha k)_+ = 0$.

Now we note (see for example Revuz and Yor [1999, Chapter VI, Exercise 1.22]) that if f is a strictly increasing function that can be written as the difference of two convex functions, $a > 0$, and Z_t a continuous semimartingale,

$$L^{f(a)}_t(f(Z)) = f'_+(a)L^a_t(Z),$$

where f'_+ is the right derivative of f. We apply the result with $f(k) = \alpha k$ to find $L^{\alpha k}_\sigma(M^g) = \alpha L^k_\sigma(M^f)$.

Then, combining the last two results, taking expectations, and noting that $\alpha > 1$, we see that if $\mathbb{E}[L^k_\sigma(M^g)] > 0$, so that $\mathbb{P}[M^g_\sigma > k] > 0$, then

$$\mathbb{E}[L^k_\sigma(M^g)] > \mathbb{E}\left[L^{\alpha k}_\sigma(M^g)\right] = \alpha\mathbb{E}[L^k_\sigma(M^f)] > \mathbb{E}[L^k_\sigma(M^f)].$$

This gives (3.1), but we require a different argument for (3.2), so suppose now that $k \leq h^g < h^f$. We have $M^g_u = h^g e^{Y_u}$, for $Y_u = \beta W_u - \frac{1}{2}\beta^2 u$, so using the local time result above, $L^k_\sigma(M^g) = kL^{\log\frac{k}{h^g}}_\sigma(Y)$, and similarly $L^k_\sigma(M^f) = kL^{\log\frac{k}{h^f}}_\sigma(Y)$. This means that our problem is equivalent to considering the local time spent by Brownian motion with drift at two different levels. For any σ such that $\mathbb{E}[L^k_\sigma(M^g)] > 0$ we have $\mathbb{P}\left(Y_\sigma < \log\frac{k}{h^g}\right) > 0$, and therefore

$$\begin{aligned}\frac{1}{2}\mathbb{E}\left[L^{\log\frac{k}{h^g}}_\sigma(Y)\right] &= \mathbb{E}\left[\left(Y_\sigma - \log\frac{k}{h^g}\right)_+\right] + \log\frac{k}{h^g}\\ &> \mathbb{E}\left[\left(Y_\sigma - \log\frac{k}{h^f}\right)_+\right] + \log\frac{k}{h^f}\\ &= \frac{1}{2}\mathbb{E}\left[L^{\log\frac{k}{h^f}}_\sigma(Y)\right].\end{aligned}$$

We now have that

$$\mathsf{SG} \supseteq \{((f,s),(g,t)) : h^f > h^g \geq k \text{ or } h^f < h^g \leq k \text{ and } \mathbb{E}[L^k_\sigma(M^g)] > 0 \text{ for all } \sigma\}$$

and the pairs in $\{((f,s),(g,t)) : h^f > h^g \geq k \text{ or } h^f < h^g \leq k\}$ that are not in SG are those for which we can find a stopping time such that the expected values of the local times at k up to the stopping time of the two processes are equal. However we have shown that if this is the case (and $s \neq t$) then these expected values must be equal to zero. This tells us that when we set our paths off at h^f and h^g, they never reach k, and so in particular $\operatorname{sgn}(M^f_\sigma - k) = \operatorname{sgn}(h^f - k)$ and $\operatorname{sgn}(M^g_\sigma - k) = \operatorname{sgn}(h^g - k)$, and this also holds for all times up to σ (and similarly when $h^g = k$). We now define our secondary optimality problem as in (1.4) with

$$\tilde{\gamma}(f,s) = -((h(f(s),s) - k)_+)^2 + ((h(f(s),s) - k)_-)^2$$

Consider a pair of paths $((f,s),(g,t))$ and a stopping time σ such that $h^f < h^g \leq k$ and $L^k_\sigma(M^f) = L^k_\sigma(M^g) = 0$. Substituting these into (1.7) gives

$$\mathbb{E}[(k - M^f_\sigma)^2] + (k - h^g)^2 < \mathbb{E}[(k - M^g_\sigma)^2] + (k - h^f)^2$$

which, by Itô-Tanaka, simplifies to

$$\mathbb{E}[\langle M^f\rangle_\sigma] < \mathbb{E}[\langle M^g\rangle_\sigma].$$

This is true since $h^f < h^g$, and we finally have that

$$\begin{aligned}
\mathsf{SG}_2 \supseteq \Big\{((f,s),(g,t)) : f(s) = g(t),\ s < t \leq \frac{2}{\beta^2}(\beta f(s) - \log(k)) \\
\text{or } s > t \geq \frac{2}{\beta^2}(\beta f(s) - \log(k))\} \\
= \{((f,s),(g,t)) : f(s) = g(t),\ h(f(s),s) < h(g(t),t) \leq k \\
\text{or } h(f(s),s) > h(g(t),t) \geq k\}.
\end{aligned}$$

Hence, by Beiglböck et al. [2017b], there exists a γ-monotone set $\Gamma \in \mathsf{S}$ such that $\mathbb{P}[((W_s)_{s\leq\tau_R}, \tau_R) \in \Gamma] = 1$, and we can complete our proof.

We know that there is a maximiser, τ_R of P_γ and $P_{\tilde{\gamma}|\gamma}$, and that we can pick a γ-

monotone set $\Gamma \in \mathsf{S}$ supporting τ_R. Define

$$\begin{aligned}
\underline{\mathcal{R}}_{CL} &:= \{(m,x) : \exists (g,t) \in \Gamma,\ h(g(t),t) \le m \le k, g(t) = x\} \\
\underline{\mathcal{R}}_{OP} &:= \{(m,x) : \exists (g,t) \in \Gamma,\ h(g(t),t) < m \le k, g(t) = x\} \\
\overline{\mathcal{R}}_{CL} &:= \{(m,x) : \exists (g,t) \in \Gamma,\ h(g(t),t) \ge m \ge k, g(t) = x\} \\
\overline{\mathcal{R}}_{OP} &:= \{(m,x) : \exists (g,t) \in \Gamma,\ h(g(t),t) > m \ge k, g(t) = x\}
\end{aligned}$$

and write $\mathcal{R}_{OP} := \underline{\mathcal{R}}_{OP} \cup \overline{\mathcal{R}}_{OP}$ and $\mathcal{R}_{CL} := \underline{\mathcal{R}}_{CL} \cup \overline{\mathcal{R}}_{CL}$. Denote the corresponding hitting times (by $(M_t(\omega), W_t(\omega))$) of these sets by $\tau_{OP} := \overline{\tau}_{OP} \wedge \underline{\tau}_{OP}$, and similarly for τ_{CL}. Note that by the form of M_t there is a one-to-one correspondence between (M_t, W_t) and (t, W_t), so we can similarly define these stopping times as hitting times of (t, W_t). We claim that $\tau_{CL} \le \tau_R \le \tau_{OP}$, and indeed we immediately see that by the definition of $\mathcal{R}_{CL}$ we have that $\tau_{CL} \le \tau_R$.

To show the second inequality pick ω such that $((W_s)_{s \le \tau_R(\omega)}, \tau_R(\omega)) \in \Gamma$ and assume for contradiction that $\overline{\tau}_{OP}(\omega) < \tau_R(\omega)$ (the argument for $\underline{\tau}_{OP}(\omega)$ is similar). Then $\exists s \in [\overline{\tau}_{OP}(\omega), \tau_R(\omega))$ such that $f := (W_r(\omega))_{r \le s}$ has $(h(f(s),s), f(s)) \in \overline{\mathcal{R}}_{OP}$. Since $s < \tau_R(\omega)$ we know that $f \in \Gamma^<$. But then by the definition of $\overline{\mathcal{R}}_{OP}$, $\exists (g,t) \in \Gamma$ such that $f(s) = g(t)$ and $h(g(t),t) > h(f(s),s) > k$ which contradicts the γ-monotonicity of Γ, since $(g(t), f(s)) \in \mathsf{SG}_2 \cap (\Gamma^< \times \Gamma)$.

Finally, by the Strong Markov Property and the fact that one-dimensional Brownian Motion immediately returns to its starting point, observe that $\underline{\tau}_{CL} = \underline{\tau}_{OP}$. To show that $\overline{\tau}_{CL} = \overline{\tau}_{OP}$ we argue as in the Rost embedding case of Beiglböck et al. [2017b, Theorem 2.4].

It is clear that we then have such a domain consisting of a barrier and an inverse barrier seperated by $K(x)$, since when $f(s) = g(t)$ we have that $h^f > h^g \implies s < t$. □

Remark 3.3. To repeat these arguments in the framework of Assumption 2 we have $h(x,t) = x^\beta \exp(-\frac{\beta(\beta-1)}{2}t)$, and instead of M^f and M^g we look at

$$\begin{aligned}
X_u^f &:= h(f(s) + X_u, s+u) = (f(s) + X_u)^\beta \exp\left(-\frac{\beta(\beta-1)}{2}(s+u)\right) \\
X_u^g &:= h(g(t) + X_u, t+u) = (g(t) + X_u)^\beta \exp\left(-\frac{\beta(\beta-1)}{2}(t+u)\right)
\end{aligned}$$

where $(X_u)_u$ is a GBM. For the inverse-barrier argument we have that, since $f(s) = g(t)$, $X_u^g = \alpha X_u^f$ for $\alpha = \exp(-\frac{\beta(\beta-1)}{2}(t-s))$. For $k \le h^g < h^f$ we write $X_u^g = h^g \mathrm{e}^{Y_u}$ where Y_u is again a martingale with a negative drift. We can then repeat exactly the arguments above.

Remark 3.4. The stop-go arguments above show that we can never embed mass along the curve K, and this can be seen as follows. If $l(x) = K(x)$ or $r(x) = K(x)$ for some x, then by the form of SG_2 we must have $l(x) = r(x) = K(x)$, as otherwise we have a stop-go pair. Take any optimiser $\tau_{\mathcal{R}}$ with corresponding l and r and let $x^* = \inf\{x > 0 : l(x) = r(x)\}$, $x_* = \sup\{x < 0 : l(x) = r(x)\}$ (with $\inf \varnothing = \infty$, $\sup \varnothing = -\infty$). Then clearly we cannot embed any mass outside of $[x_*, x^*]$. Also, we can only embed along K if $(W_{\tau_{\mathcal{R}}}, \tau_{\mathcal{R}}) \in \{(x^*, K(x^*)), (x^*, K(x^*))\}$, but this is a null event, so no optimiser can embed any mass along the curve K.

3.2.2 Non-uniqueness

We have proven that there is a solution to (BMOptSEP) which maximises our expected terminal payoff and is the hitting time of a K-cave barrier, but it is important to note that there is not a unique solution to (SEP) of this form for non-trivial distributions.

One example of non-uniqueness is a result of having a non-increasing left-hand boundary l. In this case there can be areas of l that we do not hit, and so these parts of l could actually take any form, as long as they do not embed any mass. Any choice of l has an increasing equivalent (where on any regions we do not hit, l just remains constant), and to remove this form of non-uniqueness we can assume that we are taking this choice of the left boundary. This is equivalent to the idea of uniqueness of regular barriers, as introduced in Loynes [1970].

Even once we have made this choice of l, a more troublesome form of non-uniqueness can occur. Consider for example an atomic distribution with atoms at three points N, $-N$, and $z \in (0, N)$. Corresponding K-cave stopping regions will have absorbing barriers at $\pm N$ (to ensure the stopped process is uniformly integrable), and two barriers with end points $l(z)$ and $r(z)$ at z. Suppose this stopping region stops mass at $N, -N, z$ with probabilities p_N, p_{-N}, p_z, respectively.

Then these probabilities must sum to one, and we also have the martingale condition, so

$$
\begin{aligned}
p_N + p_{-N} + p_z &= 1 \\
Np_N - Np_{-N} + zp_z &= 0.
\end{aligned}
$$

These two equations fix p_N, p_{-N} for a given p_z.

We can change our stopping time by moving the points $r(z)$ and $l(z)$, and it is easy to see that increasing $l(z)$ (moving our left hand boundary at z to the right) increases p_z. Similarly, decreasing $l(z)$ decreases the amount of paths stopped by this boundary, so decreases p_z, and moving $r(z)$ has the opposite effect.

Therefore, writing $p_z = p_z(l,r)$ as a function of $l := l(z)$ and $r := r(z)$, p_z is increasing in $l(z)$ and decreasing in $r(z)$.

Suppose we have some $l(z)$ and $r(z)$ such that $p_z(l,r) = \mu(\{z\})$ (and therefore $p_N = \mu(\{N\})$ and $p_{-N} = \mu(\{-N\})$). Consider increasing $l(z)$ by some small amount, ε, to a new value $\tilde{l}(z) = l(z) + \varepsilon$, so $p_z(l+\varepsilon, r) > p_z(l,r) = \mu(\{z\})$. For certain distributions μ, we will then be able to increase $r(z)$ to some new $\tilde{r}(z) = r(z) + \delta$, for $\delta > 0$, such that $p_z(l+\varepsilon, r+\delta) = p_z(l,r) = \mu(\{z\})$.

For 'nice' μ (with atom at z not too small or too large), we could find that there is a non-trivial interval (a,b) and a function $\mathsf{r} : (a,b) \to [K(z), \infty)$ such that $p_z(l, \mathsf{r}(l)) = \mu(\{z\})$ for any $l \in (a,b)$, so we have infinitely many K-cave barriers that embed μ. We then need to move l inside the interval (a,b) to get different embeddings and find the optimal such stopping time.

In less trivial cases we could have multiple barriers embedding μ each with a different stopping time, and therefore a different payoff, so we would like to find a condition on the barriers that determines whether a given K-cave barrier which embeds the correct law is also optimal. This is the aim of the following section, where we consider the dual problem.

3.3 Heuristic PDE Arguments for Duality

In this section, we take a heuristic approach to the dual problem. Our aim is to establish a condition on the barriers which will correspond to a form of dual attainment. Very loosely, our primal problem can be reconsidered in the framework of optimising over the class of K-cave barriers which embed the desired law. Assuming that an appropriate dual problem can be formulated, one might expect to be able to characterise optimality in terms of dual attainment of a corresponding dual solution. The aim of this section is to construct a candidate dual solution, and provide a condition for feasibility. In subsequent sections, we will justify the condition, by showing both that there exist K-cave barriers satisfying the condition, and that the condition is sufficient for optimality.

The following analysis is motivated by Henry-Labordère, Section 4, and relies on PDE arguments. In this section, our arguments are purely formal, and aim to provide justification for our later results. The dual problem has a natural interpretation as a super-hedging problem, and we use that language here, although it can also be understood in terms of martingale arguments.

Suppose we want to superhedge the option with payoff $F(x,t)$, and to make use of our LETF motivation we work under the conditions of Assumption 2, so we assume that X_t is a Geometric Brownian Motion and $F = F_L$ (the argument is easily transferrable to Assumption 1).

For our superhedging portfolio we wish to hold a static portfolio of call options with price process $\lambda(X_t)$, and trade a dynamic portfolio with value $\gamma(X_t,t)$ such that $F(X_t,t) \leq \lambda(X_t) + \gamma(X_t,t)$ at all times t. The dynamic portfolio value, $\gamma(X_t,t)$, can be viewed as the gains from trading, and we might therefore expect it to be a martingale, or more generally a supermartingale by allowing ourselves to withdraw positive amounts from the balance.

Portfolios satisfying these conditions can be thought of as feasible superhedging (dual) portfolios. The initial cost of setting up this portfolio is then an upper bound on the arbitrage-free price of the option with payoff F. The aim of the heuristic arguments in this section is to find a condition under which the superhedging portfolio gives the *least* upper bound. In the case of the Root and Rost embeddings it is shown in Cox and Wang [2013b,a] that, under certain assumptions, the necessary conditions are that $\gamma(X_{t\wedge\tau}, t\wedge\tau)$ should be a martingale, and that our superhedge should be an exact hedge in the stopping region. The time 0 cost of setting up this portfolio is the least upper bound on the price of the option, and therefore is equal to the value of our primal problem.

We follow the same idea here, attempting to construct a superhedging portfolio that has these properties. In the case of the Root and Rost embeddings (for certain payoffs) we can always find a portfolio that replicates the value of the option, however in the K-cave case we require an extra condition to ensure that our chosen portfolio is indeed optimal, and this section motivates the form of this condition.

Initially we choose some region $\mathcal{D}$ (which will correspond to $\mathcal{R}^{\complement}$), and a function $\lambda(x)$ representing a static portfolio of call options at all strikes. Let $\tau_{\mathcal{D}}$ be the exit time of $\mathcal{D}$, and $\tau_{\mathcal{D}}^{(x,t)}$ be the first hitting time of the stopping region for a Brownian motion setting off from (x,t), so $\tau_{\mathcal{D}} \equiv \tau_{\mathcal{D}}^{(0,0)}$. Then

$$\begin{aligned}\tau_{\mathcal{D}} &:= \inf\{t \geq 0 :\ (W_t, t) \notin \mathcal{D}\},\\ \tau_{\mathcal{D}}^{(x,t)} &:= t + \inf\{s \geq 0 :\ (x + W_s, t+s) \notin \mathcal{D}\}.\end{aligned}$$

For any function f we write $\mathbb{E}^{x,t}\left[f(X_{\tau_{\mathcal{D}}}, \tau_{\mathcal{D}})\right]$ to mean $\mathbb{E}\left[f\left(X_{\tau_{\mathcal{D}}^{(x,t)}}, \tau_{\mathcal{D}}^{(x,t)}\right)\right]$.

We set our dynamic trading strategy to be

$$\gamma(x,t) := \begin{cases} F^{\lambda}(x,t) & \text{for}(x,t) \notin \mathcal{D} \\ \mathbb{E}^{x,t}[F^{\lambda}(X_{\tau_{\mathcal{D}}}, \tau_{\mathcal{D}})] & \text{for}(x,t) \in \mathcal{D} \end{cases}$$

where $F^{\lambda}(x,t) := F(x,t) - \lambda(x)$. Then for $\gamma(x,t) + \lambda(x)$ to be a superhedge (in the above sense), we require

$$\mathcal{L}\gamma := \frac{x^2}{2}\partial_x^2\gamma + \partial_t\gamma \leq 0 \qquad \forall(x,t) \tag{3.3}$$

$$\gamma \geq F^{\lambda} \quad \forall(x,t). \tag{3.4}$$

We can see immediately that (3.3) holds with equality in $\mathcal{D}$, and (3.4) holds with equality in $\mathcal{D}^{\complement}$.

Consider a domain $\mathcal{D}$ which is the continuation region of a K-cave barrier, so for $(x,t) \in \mathcal{D}$ we have that $l(x) := \inf\{s < t :\ (x,s) \in \mathcal{D}\}$ and $r(x) := \sup\{s > t :\ (x,s) \in \mathcal{D}\}$ are independent of t (otherwise we contradict the (inverse) barrier properties of the region). We want our superhedge to match the payoff on the boundary, so we require

$$\begin{aligned}\gamma(x, l(x)) &= F^{\lambda}(x, l(x))\\ \gamma(x, r(x)) &= F^{\lambda}(x, r(x)).\end{aligned} \tag{3.5}$$

Then we wish to find $\mathcal{D}$, λ such that

$$\begin{aligned}\mathcal{L}\gamma &= 0 && \text{in } \mathcal{D}\\ \gamma &= F^{\lambda} && \text{on } \partial\mathcal{D}.\end{aligned}$$

Note that with this boundary condition and sufficient smoothness we find $\partial_t\gamma = \partial_t F^{\lambda} =$

$\partial_t F$ on $\partial\mathcal{D}$, and then, writing $\eta = \partial_t \gamma$, we expect

$$\begin{aligned} \mathcal{L}\eta &= 0 && \text{in } \mathcal{D} \\ \eta = \partial_t F^\lambda &= \partial_t F && \text{on } \partial\mathcal{D}. \end{aligned}$$

We can then use Dynkin's Formula to deduce that

$$\eta(x,t) = \mathbb{E}^{x,t}[\partial_t F(X_{\tau_\mathcal{D}}, \tau_\mathcal{D})] =: M(x,t) \tag{3.6}$$

and so

$$\gamma(x,t) = -\int_t^{r(x)} M(x,v)\mathrm{d}v - \xi(x) \tag{3.7}$$

where $\xi(x)$ is some function, which we will choose to ensure $\mathcal{L}\gamma = 0$. We could take any upper limit in the integral, but we will see later that $r(x)$ is a natural choice.

With this form for the function γ, we can consider the boundary conditions, (3.5). Rearranging (3.5), we see that we must have

$$\lambda(x) = F(x, l(x)) - \gamma(x, l(x)) = F(x, r(x)) - \gamma(x, r(x))$$

Observing that $F(x, r(x)) = 0$ (by the form of F, since $r(x) \geq K(x)$), we note that this holds whenever

$$\Gamma(x) := F(x, l(x)) + \int_{l(x)}^{r(x)} M(x,v)\,\mathrm{d}v = 0 \quad \forall x \in \mathcal{D}. \tag{3.8}$$

More generally, if we only require that (3.4) holds, a necessary condition on the boundary is that

$$\lambda(x) \geq \max\{F(x, l(x)) + \int_{l(x)}^{r(x)} M(x,v)\,\mathrm{d}v + \xi(x), \xi(x)\},$$

and we see that if $\Gamma(x) = 0$, it is sufficient to take $\xi(x) = \lambda(x)$. Since ξ was chosen to make $\mathcal{L}\gamma = 0$, this will effectively fix λ. In the next section we will see that it is sufficient for (3.4) to hold on the boundaries in order to deduce that it holds in the interior as well.

Then, to summarise this section, given a set $\mathcal{D}$ which is the continuation region of a K-cave barrier, we (heuristically) can construct functions $\gamma_\mathcal{D}$ (given by (3.7)) and $\lambda_\mathcal{D}(x) := \max\{F(x, l(x)) + \int_{l(x)}^{r(x)} M(x,v)\,\mathrm{d}v + \xi(x), \xi(x)\}$ such that (3.3) and (3.4) hold. If in addition τ is a (uniformly integrable) stopping time such that $X_\tau \sim \mu$, then:

$$\begin{aligned}\mathbb{E}\left[F(X_\tau, \tau)\right] &\leq \mathbb{E}\left[\gamma_{\mathcal{D}}(X_\tau, \tau) + \lambda_{\mathcal{D}}(X_\tau)\right] \\ &\leq \gamma_{\mathcal{D}}(X_0, 0) + \int \lambda_{\mathcal{D}}(x)\mu(\mathrm{d}x),\end{aligned}$$

and therefore

$$\sup_{\tau: X_\tau \sim \mu} \mathbb{E}\left[F(X_\tau, \tau)\right] \leq \inf_{\mathcal{D}} \left\{\gamma_{\mathcal{D}}(X_0, 0) + \int \lambda_{\mathcal{D}}(x)\mu(\mathrm{d}x)\right\}. \tag{3.9}$$

Moreover, if $\mathcal{D}$ is such that $X_{\tau_{\mathcal{D}}} \sim \mu$, $\gamma_{\mathcal{D}}(X_t \wedge \tau_{\mathcal{D}}, t \wedge \tau_{\mathcal{D}})$ is a martingale, and $\Gamma(x) = 0$, the inequalities above are equalities for $\tau_{\mathcal{D}}$, and so the supremum and the infimum coincide.

Our aim in the next section will be to make these heuristic arguments rigorous whilst showing that, in fact, any set $\mathcal{D}$ which is the continuation region of a K-cave barrier embedding μ *and* which satisfies (3.8) (or a slightly refined version of (3.8)) gives equality in (3.9).

3.4 Optimality

We have introduced the dual problem of choosing a K-cave barrier which embeds μ and such that $\Gamma(x) = 0$. In this section we will make these heuristic arguments rigorous, and show that if we have a K-cave barrier that satisfies these conditions, then it does indeed give rise to an optimal embedding. We will modify the arguments presented in Cox and Wang [2013a], using the heuristics of the previous section to motivate our choice of functions, but writing our problem under Assumption 1, although an essentially identical analysis holds under Assumption 2. Hence, for a Brownian motion W, we wish to find an embedding τ of the form given in Theorem 3.2 (the hitting time of a K-cave barrier), and functions $G(x, t)$ and $H(x)$ such that

- $F(x, t) \leq G(x, t) + H(x)$ everywhere (3.10)
- $G(W_t, t)$ is a supermartingale (3.11)
- $F(W_\tau, \tau) = G(W_\tau, \tau) + H(W_\tau)$ (3.12)
- $G(W_{t\wedge\tau}, t \wedge \tau)$ is a martingale. (3.13)

We use the previous section to motivate a possible form of our super-replicating portfolio, and we will see that it is highly dependent on the region $\mathcal{D}$. The idea here is that the portfolio we propose, which depends heavily on the stopping region, is 'dual feasible' for any stopping region, and then the correct choice of our region $\mathcal{D}$, or equivalently our curves l, r, will correspond to satisfying the complementary slackness conditions of our primal-dual problem. The conditions (3.10) and (3.11) are our dual conditions, i.e. our dual problem is to minimise $\mathbb{E}\left[G(W_\tau,\tau)+H(W_\tau)\right]$ over functions G, H such that (3.10), (3.11) hold. Then (3.12) and (3.13) are the complementary slackness conditions. In Section 2 we prove that our choices of G, H are indeed the correct ones, so the condition we give is both necessary and sufficient.

Consider a K-cave barrier $\mathcal{R}$ with continuation region $\mathcal{D}=\mathcal{R}^{\complement}$. Let $\tau_{\mathcal{D}}$ be the exit time of $\mathcal{D}$, and $\tau_{\mathcal{D}}^{(x,t)}$ be the first hitting time of the stopping region for a Brownian motion setting off from (x,t), so $\tau_{\mathcal{D}}\equiv\tau_{\mathcal{D}}^{(0,0)}$. Then

$$\begin{aligned}\tau_{\mathcal{D}} &:= \inf\{t\geq 0: t\notin (l(W_t),r(W_t))\},\\ \tau_{\mathcal{D}}^{(x,t)} &:= t+\inf\{s\geq 0:\ (x+W_s,t+s)\notin\mathcal{D}\}.\end{aligned}$$

Recall that $F(x,t)=(h(x,t)-k)_+$, for $h(x,t)=\exp(\beta x-\frac{1}{2}\beta^2 t)$, so for $t\neq K(x)$, the time derivative of F is

$$\partial_t F(x,t)=\partial_t h(x,t)\mathbf{1}\{h(x,t)>k\}=\partial_t h(x,t)\mathbf{1}\{t\leq K(x)\}=-\frac{\beta^2}{2}h(x,t)\mathbf{1}\{t<K(x)\}.$$

Note that F is not differentiable across K. The shape of the optimal K-cave barrier means that we never embed mass along K (see Remark 3.4), and so this is not important to us, we can consider either the left or right derivative at $t=K(x)$ in order to define $\partial_t F(x,t)$ everywhere.

Looking at (3.6) and (3.7), we define

$$\begin{aligned} &G(x,t) := G^*(x,t)-Z(x),\\ \text{where}\qquad &G^*(x,t) := -\int_t^{r(x)} M(x,s)\mathrm{d}s,\\ &M(x,t) := \mathbb{E}^{x,t}\left[\partial_t F\left(W_{\tau_{\mathcal{D}}},\tau_{\mathcal{D}}\right)\right]=-\frac{\beta^2}{2}\mathbb{E}^{x,t}\left[h(W_{\tau_{\mathcal{D}}},\tau_{\mathcal{D}})\mathbf{1}\{\tau_{\mathcal{D}}<K(W_{\tau_{\mathcal{D}}})\}\right],\end{aligned}$$

and $Z(x)$ is chosen as above to ensure that $G(W_t,t)$ is a martingale in $\mathcal{D}$. Here we write $\mathbb{E}^{x,t}\left[\partial_t F\left(W_{\tau_\mathcal{D}}, \tau_\mathcal{D}\right)\right]$ to mean

$$\mathbb{E}\left[\partial_t F\left(W_{\tau_\mathcal{D}^{(x,t)}}, \tau_\mathcal{D}^{(x,t)}\right)\right] = \mathbb{E}[\partial_t F\left(W_{\tau_\mathcal{D}}, \tau_\mathcal{D}\right)|\, W_t = x, t \leq \tau_\mathcal{D}].$$

In particular, in the continuation region $M(W_t,t) = \mathbb{E}\left[\partial_t F\left(W_{\tau_\mathcal{D}}, \tau_\mathcal{D}\right)|\, \mathcal{F}_t\right]$ and $M(W_t,t)$ is therefore a martingale. We have taken the Brownian motion payoff, $F = F_{BM}$, and the only difference in the case of the LETF payoff is that $\frac{\beta^2}{2}$ becomes $\frac{\beta(\beta-1)}{2}$.

Since $h(W_t,t)$ is a non-negative martingale,

$$M(x,t) - \frac{\beta^2}{2}\mathbb{E}^{x,t}\left[h(W_{\tau_\mathcal{D}}, \tau_\mathcal{D})\mathbf{1}\{\tau_\mathcal{D} > K(W_{\tau_\mathcal{D}})\}\right] = -\frac{\beta^2}{2}h(x,t)$$

and then we have

$$\begin{aligned} M(x,t) &= -\frac{\beta^2}{2}h(x,t) \quad \text{for } (x,t) \in \{(x,t) : t \leq l(x)\} \\ -\frac{\beta^2}{2}h(x,t) \leq M(x,t) &\leq 0 \quad \text{for } (x,t) \in D = \{(x,t) : l(x) \leq t \leq r(x)\} \\ M(x,t) &= 0 \quad \text{for } (x,t) \in \{(x,t) : t \geq r(x)\}. \end{aligned}$$

Define μ_l and μ_r to be the the distributions of the mass embedded along $l(x)$ and $r(x)$ respectively, that is for any $A \subseteq \mathbb{R}$,

$$\begin{aligned} \mu_l(A) &:= \mathbb{P}\left(W_{\tau_\mathcal{D}} \in A, W_{\tau_\mathcal{D}} \leq l(\tau_\mathcal{D})\right) \\ \mu_r(A) &:= \mathbb{P}\left(W_{\tau_\mathcal{D}} \in A, W_{\tau_\mathcal{D}} \geq r(\tau_\mathcal{D})\right). \end{aligned}$$

We say that both barriers are *attainable* at x if $x \in \text{supp}(\mu_l) \cap \text{supp}(\mu_r)$. Define

$$\Gamma(x) := F(x, l(x)) + \int_{l(x)}^{r(x)} M(x,v)\text{d}v.$$

From the heuristics in the previous section, we propose the following condition on our barriers l and r for optimality:

$$\begin{aligned} \Gamma(x) &\geq 0 \quad \mu_l\text{-a.s.} \\ \Gamma(x) &\leq 0 \quad \mu_r\text{-a.s.} \end{aligned} \tag{Γ}$$

Theorem 3.5. *If $\mathcal{R}$ is a K-cave barrier that embeds a distribution μ and also satisfies* (Γ)*, then $\tau_\mathcal{D}$ is optimal.*

To show this we first need to show that our function G^* is such that we can choose Z and H to give the required properties. First, let $x^* := \inf\{x > 0 : \ l(x) = K(x) = r(x)\}$, where we set $\inf \emptyset = \infty$ if our barriers never meet. Note that if $x^* < \infty$, then our distribution μ embeds no mass above x^* and so any pair of barriers embedding μ must meet at x^*, i.e. $l(x^*) = r(x^*) = K(x^*)$. If this wasn't the case then with positive probability we can find $0 < t < \tau_{\mathcal{D}}$ such that $W_t = x > x^*$. Then $\tau_{\mathcal{D}}^{(x,t)} \geq t + H_{x-x^*}(W)$, and in particular $\mathbb{E}\left[\tau_{\mathcal{D}}^{(x,t)}\right] = \infty$, so $(W_{t\wedge\tau_{\mathcal{D}}})_t$ is not uniformly integrable. Therefore our process is always stopped below this point, or before $H_{x^*} = \inf\{t \geq 0 : \ W_t = x^*\}$.

Lemma 3.6. *We can find a function Z such that the process*

$$G(W_{t\wedge\tau_{\mathcal{D}}}, t \wedge \tau_{\mathcal{D}}) \qquad \textit{is a martingale,}$$

and

$$G(W_t, t) \qquad \textit{is a supermartingale up to } H_{x^*}.$$

Proof. We first show that we can find an increasing process $A_t = A(W_t)$, depending only on W_t, such that $G^*(W_t, t) - A_t$ is a martingale in $\mathcal{D}$, and a supermartingale in general. We note that, for either of our payoffs, $h(x,t) < \infty$ for any (x,t) and h is integrable on any $[y,z] \times [0,\infty)$ for any bounded $y < z$. This means that $|G^*|$ is bounded on compact sets in space for all $t \geq 0$, and so all of the terms in the following arguments are well defined. In much of what follows we will take our process at some point (W_t, t) and consider letting it run until some stopping time, perhaps $\tau := \inf\{u > 0 : \ |W_{t+u} - W_t| \geq \delta\} \wedge \epsilon$ for some small δ and ϵ.

1. *Show $G^*(W_t,t)$ is a submartingale in $\mathcal{D}$*: First take $(W_t, t) \in \mathcal{D}$, the continuation region, and τ a stopping time of the above form such that $t + \tau < \tau_{\mathcal{D}}$, so we remain in the continuation region. Then,

$$\begin{aligned}
\mathbb{E}[G^*(W_{t+\tau}, t+\tau) - &G^*(W_t,t)|\mathcal{F}_t] \\
&= \mathbb{E}\left[-\int_{t+\tau}^{r(W_{t+\tau})} M(W_{t+\tau}, u)\mathrm{d}u + \int_t^{r(W_t)} M(W_t,u)\mathrm{d}u \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[-\int_t^{r(W_t)} \Big(M(W_{t+\tau}, u+\tau) - M(W_t,u)\Big)\mathrm{d}u \middle| \mathcal{F}_t\right] \\
&\qquad + \mathbb{E}\left[\int_{r(W_{t+\tau})}^{r(W_t)+\tau} M(W_{t+\tau},u)\mathrm{d}u \middle| \mathcal{F}_t\right].
\end{aligned}$$

It is natural to split the integrals up in this way since we know that in the continuation region M is a martingale, and so we hope to use Fubini and the martingale property to argue that the first term is zero. However, our Brownian motion does not stay within $\mathcal{D}$ for all $u \in (t, r(W_t))$, as shown in Figure 3-3, and so we cannot use this martingale property and instead must argue about the sign of this term. Since we have assumed that $t + \tau < \tau_{\mathcal{D}}$, we know that the path leaving (W_t, t) does not cross the the left hand boundary l. Then when we move the starting the point to (W_t, u) for $u \geq t$, by the definition of the inverse barrier shape, we know that these paths also cannot cross l, however they may now cross r. Also note that $t < r(W_t)$ since we are in $\mathcal{D}$.

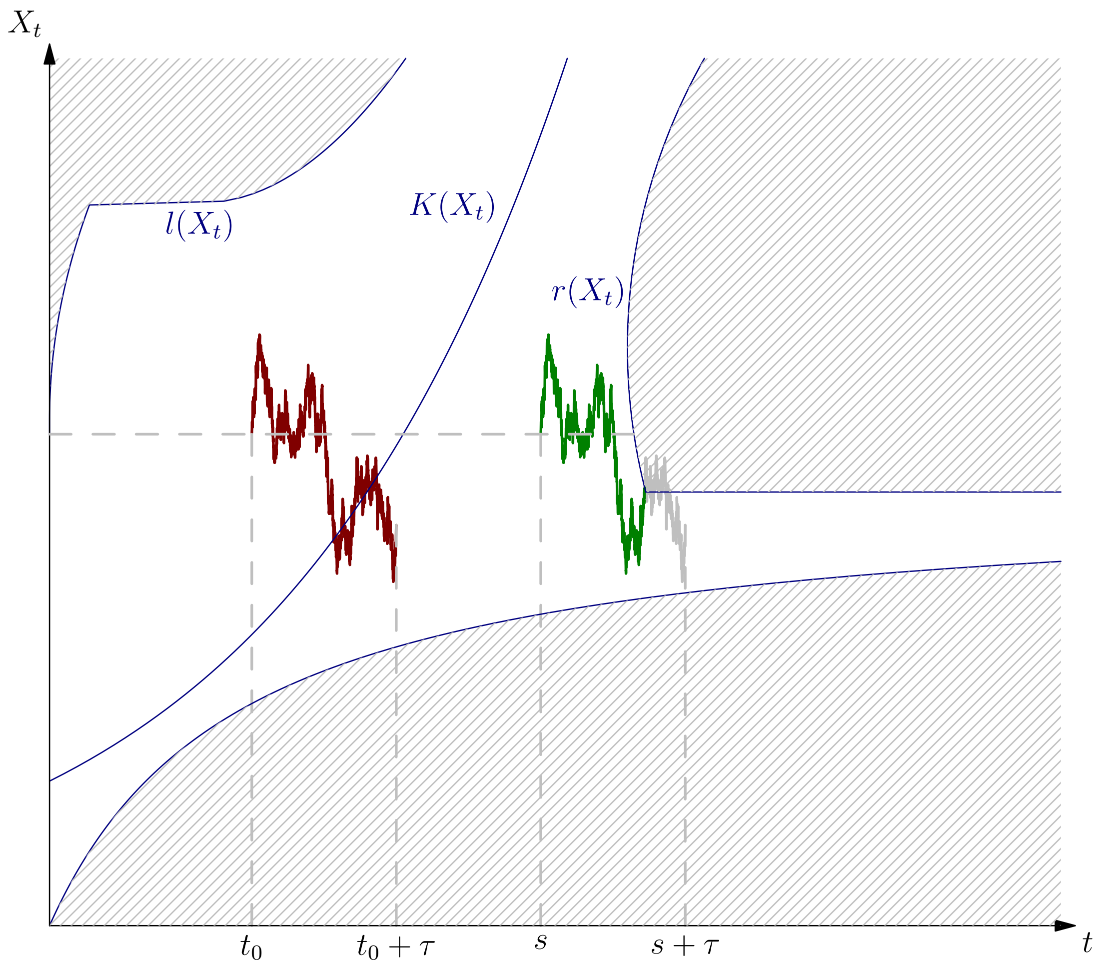

Figure 3-3: Here we have a path leaving from (W_{t_0}, t_0) running for a time τ inside $\mathcal{D}$ and we consider moving this path along the time axis, so we may now exit $\mathcal{D}$.

Note that $M(W_{t+\tau}, u) = 0$ if $u \geq r(W_{t+\tau})$, so the final term is 0 if $r(W_{t+\tau}) \leq r(W_t)$. We also know that M is everywhere non-positive, so if $r(W_{t+\tau}) \geq r(W_t)$ then the final integral is non-negative. Therefore, the last term in the above is always non-negative.

We can also show that the other term in the above expression is non-negative. Recall that $\tau_{\mathcal{D}}^{(x,t)} := t + \inf\{s \geq 0 : (x + W_s, t + s) \notin \mathcal{D}\} = t + \inf\{s \geq 0 : u + s \geq r(W_{t+s})\}$, where the second equality now follows since we cannot cross l into the inverse barrier. Take $u \in (t, r(W_t))$ and let $\hat{\tau}_{\mathcal{D}} := (\tau_{\mathcal{D}}^{(W_t,u)} - u) \wedge \tau$. When $\hat{\tau}_{\mathcal{D}} = \tau$ we have $M(W_{t+\tau}, u + \tau) = M(W_{t+\hat{\tau}_{\mathcal{D}}}, u + \hat{\tau}_{\mathcal{D}}) \leq 0$, and when $\hat{\tau}_{\mathcal{D}} < \tau$ we have that $M(W_{t+\tau}, u + \tau) \leq 0 = M(W_{t+\hat{\tau}_{\mathcal{D}}}, u + \hat{\tau}_{\mathcal{D}})$. Therefore,

$$\mathbb{E}\left[M(W_{t+\tau}, u + \tau)|\mathcal{F}_t\right] \leq \mathbb{E}\left[M(W_{t+\hat{\tau}_{\mathcal{D}}}, u + \hat{\tau}_{\mathcal{D}})|\mathcal{F}_t\right] = M(W_t, u)$$

since $M(W_t, t)$ is a martingale in $\mathcal{D}$. Swapping the expectation and the integral by Tonelli's theorem, we conclude that

$$\mathbb{E}\left[-\int_t^{r(W_t)} \Big(M(W_{t+\tau}, u + \tau) - M(W_t, u)\Big)\mathrm{d}u\Big|\mathcal{F}_t\right] \geq 0. \tag{3.14}$$

Provided we have integrability, this tells us that $G^*(W_t, t)$ is a submartingale in $\mathcal{D}$, and therefore the Doob-Meyer Decomposition Theorem tells us that there exists a unique, increasing, predictable process A_t such that $M_t = G^*(W_t, t) - A_t$ is a martingale in $\mathcal{D}$. But,

$$\mathbb{E}\left[|G^*(W_t, t)|\right] \leq \mathbb{E}\left[-\int_0^{r(W_t)} M(W_t, s)\mathrm{d}s\right] \leq \mathbb{E}\left[F(W_t, 0)\right] < \infty \quad \forall t$$

for either of our payoffs, and so we have integrability.

2. *A_t depends only on W_t*: To think more about A_t we consider, as usual, a time $t < \tau_{\mathcal{D}}$ and then run our process from t up until a small stopping time τ such that $t + \tau < \tau_{\mathcal{D}}$, but now we imagine moving this path along the time axis. We then have $t < \tau_{\mathcal{D}}$, $t + \tau < \tau_{\mathcal{D}}^{(W_t,t)}$ and we take $s < r(W_t)$ such that $s + \tau < \tau_{\mathcal{D}}^{(W_t,s)}$. From the definition of G^*, we have

$$\begin{aligned}\mathbb{E}\left[G^*(W_{t+\tau}, s + \tau) - G^*(W_t, s)|\mathcal{F}_t\right] &= \mathbb{E}\left[G^*(W_{t+\tau}, t + \tau) - G^*(W_t, t)|\mathcal{F}_t\right] \\ &\quad + \mathbb{E}\left[\int_t^s \left(M(W_{t+\tau}, u + \tau) - M(W_t, u)\right)\mathrm{d}u\Big|\mathcal{F}_t\right].\end{aligned} \tag{3.15}$$

Since $s < s + \tau < \tau_{\mathcal{D}}$ and $t < t + \tau < \tau_{\mathcal{D}}$, and by the shape of our boundaries, we have that $(W_t, u), (W_{t+\tau}, u + \tau) \in \mathcal{D}$ for $u \in (t, s)$, and as $M(W_t, t)$ is a martingale in $\mathcal{D}$, we have that

$$\mathbb{E}\left[M(W_{t+\tau}, u + \tau)|\mathcal{F}_t\right] = \mathbb{E}\left[M(W_t, u)|\mathcal{F}_t\right] = M(W_t, u)$$

for all $u \in (t, s)$. By Fubini the final term in (3.15) is 0, so

$$\mathbb{E}\left[G^*(W_{t+\tau}, s+\tau) - G^*(W_t, s)|\mathcal{F}_t\right] = \mathbb{E}\left[G^*(W_{t+\tau}, t+\tau) - G^*(W_t, t)|\mathcal{F}_t\right]. \tag{3.16}$$

This tells us that in $\mathcal{D}$, A_t depends only on W_t and not directly on t. If we now consider taking any s, but keeping t such that $t + \tau < \tau_{\mathcal{D}}$, then we still have (3.15), but now we can show that the final term is actually non-positive.

Since we now consider any s, we will no longer always be in the continuation region, and we need to consider crossing the boundaries. We know from Theorem 3.2 that our right-hand boundary r is a barrier, and l is an inverse barrier. If we have $t+\tau < \tau_{\mathcal{D}}^{(W_t,t)}$, then $(W_{t+u}, t+u) \in \mathcal{D}$ for every $u \in (0, \tau)$, so in particular we do not cross the left hand boundary l. If $t < s$ then, since l is an inverse barrier, we must also have that $s+u > l(W_{t+u})$ for every $u \in (0, \tau)$, so shifting this part of our path to the right cannot cause us to cross l. We can however cross r, so we need to argue exactly as with (3.14) to see that

$$\mathbb{E}\left[\int_t^s \left(M(W_{t+\tau}, u+\tau) - M(W_t, u)\right) \mathrm{d}u \Big| \mathcal{F}_t\right] \leq 0$$

and so

$$\mathbb{E}\left[G^*(W_{t+\tau}, s+\tau) - G^*(W_t, s)|\mathcal{F}_t\right] \leq \mathbb{E}\left[G^*(W_{t+\tau}, t+\tau) - G^*(W_t, t)|\mathcal{F}_t\right].$$

If we take $s < t$ then we instead have that $s + u < r(W_{t+u})$ for every $u \in (0, \tau)$, and so we do not cross r but could cross l. The argument here is similar in that we let $\tilde{\tau}_{\mathcal{D}} = (\tau_{\mathcal{D}}^{(W_t,u)} - u) \wedge \tau$, take $u \in (s, t)$ and compare $M(W_{t+\tau}, u+\tau)$ and $M(W_{t+\tilde{\tau}_{\mathcal{D}}}, u+\tilde{\tau}_{\mathcal{D}})$. On $\{\tilde{\tau}_{\mathcal{D}} = \tau\}$ we clearly have $M(W_{t+\tau}, u+\tau) = M(W_{t+\tilde{\tau}_{\mathcal{D}}}, u+\tilde{\tau}_{\mathcal{D}})$, but when $\tilde{\tau}_{\mathcal{D}} < \tau$ we have

$$\begin{aligned}
\mathbb{E}\left[M(W_{t+\tilde{\tau}_{\mathcal{D}}}, u+\tilde{\tau}_{\mathcal{D}})|\mathcal{F}_t\right] &= \mathbb{E}\left[\frac{-\beta^2}{2} h(W_{t+\tilde{\tau}_{\mathcal{D}}}, u+\tilde{\tau}_{\mathcal{D}}) \Big| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{-\beta^2}{2} h(W_{t+\tau}, u+\tau) \Big| \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[M(W_{t+\tau}, u+\tau)|\mathcal{F}_t\right]
\end{aligned}$$

by the Optional Sampling Theorem, since both our stopping times are bounded. Combining these as before and using Fubini, we again have

$$\mathbb{E}\left[\int_t^s \left(M(W_{t+\tau}, u+\tau) - M(W_t, u)\right) \mathrm{d}u \Big| \mathcal{F}_t\right] \leq 0$$

and so for $t < t + \tau < \tau_{\mathcal{D}}$ and any s, we have that

$$\mathbb{E}\left[G^*(W_{t+\tau}, s+\tau) - G^*(W_t, s)|\mathcal{F}_t\right] \leq \mathbb{E}\left[G^*(W_{t+\tau}, t+\tau) - G^*(W_t, t)|\mathcal{F}_t\right] \\ = \mathbb{E}\left[A_{t+\tau} - A_t|\mathcal{F}_t\right]. \tag{3.17}$$

3. $G(W_t, t)$ *has the desired properties*: We now combine the above two results to show that we have the supermartingale property we require, noting that we already have the martingale property in $\mathcal{D}$ as this is how we chose A. Consider now arbitrary s and τ and suppose that we can fix a t such that $(s, W_t) \in \mathcal{D}$ and $s + \tau < \tau_{\mathcal{D}}^{(W_t,s)}$. Then from (3.16) and (3.17) we have

$$\mathbb{E}\left[G^*(W_{t+\tau}, t+\tau) - G^*(W_t, t)|\mathcal{F}_t\right] \leq \mathbb{E}\left[G^*(W_{t+\tau}, s+\tau) - G^*(W_t, s)|\mathcal{F}_t\right].$$

We can use this to give the following:

$$\begin{aligned}
\mathbb{E}\left[G^*(W_{t+\tau}, t+\tau) - A_{t+\tau}|\mathcal{F}_t\right] &\leq \mathbb{E}\left[G^*(W_{t+\tau}, s+\tau) - A_{t+\tau}|\mathcal{F}_t\right] \\
&\qquad + G^*(W_t, t) - G^*(W_t, s) \\
&= G^*(W_t, t) + \mathbb{E}\left[G^*(W_{t+\tau}, s+\tau) - G^*(W_t, s)|\mathcal{F}_t\right] \\
&\qquad - \mathbb{E}\left[A_{t+\tau}|\mathcal{F}_t\right] \\
&= G^*(W_t, t) + \mathbb{E}\left[A_{t+\tau} - A_t|\mathcal{F}_t\right] - \mathbb{E}\left[A_{t+\tau}|\mathcal{F}_t\right] \\
&= G^*(W_t, t) - A_t,
\end{aligned}$$

which is exactly the supermartingale property we are looking for.

It will not always be the case that we can find such a t as above, in fact for a given t, τ we may find that $\tau > \tau_{\mathcal{D}}^{(W_t,s)} - s \; \forall s$ such that $(W_t, s) \in \mathcal{D}$. We then need to find a sequence of stopping times that sum to τ and use the above on each of the intervals. Suppose first that our curves l, r do not meet, or they do so well away from t and $t+\tau$. We can then choose some $s \in (l(W_t), r(W_t))$ (we will take $s = K(W_t)$ for simplicity) and we run the process from (W_t, s) until we hit a boundary, call this stopping time σ_1. We then move back into our continuation region and set off from $(K(W_{s+\sigma_1}), W_{s+\sigma_1})$, and run again for a time σ_2 until we hit the boundary. Provided our barriers do not meet we can continue this until we reach $s+\tau$ in a finite number of steps. We can then write $\mathbb{E}\left[G^*(W_{t+\tau}, t+\tau) - G^*(W_t, t)|\mathcal{F}_t\right]$ as a telescoping sum and show the inequality as before. From the exact argument above with τ when we do not leave the region, we have that

$$\mathbb{E}\left[G^*(W_{t+\sigma_1}, t+\sigma_1) - A_{t+\sigma_1}|\mathcal{F}_t\right] \leq G^*(W_t, t) - A_t,$$

and also

$$\mathbb{E}\left[G^*(W_{t+\sigma_{j+1}}, t+\sigma_{j+1}) - A_{t+\sigma_{j+1}}|\mathcal{F}_t\right] \leq \mathbb{E}\left[G^*(W_{t+\sigma_j}, t+\sigma_j) - A_{t+\sigma_j}|\mathcal{F}_t\right]$$

for our stopping times $\{\sigma_j\}_j$ where $\sigma_j = \tau$ for some j. We then combine these results in our telescoping sum to get the supermartingale property as before. If $W_{t+\tau} < x^*$ then we can always find a finite sequence of stopping times that sum to τ. The only other case is where $W_{t+\tau} = x^*$. In this case we again require a sequence of stopping times, but this time we will could have infinitely many, with the sum converging to τ, but then we can work as before but using Fubini to interchange our expectation and the infinite sum.

We now know that we can find an increasing process A_t, dependent only on W_t, such that $G^*(W_t, t) - A_t$ is a martingale up until $\tau_{\mathcal{D}}$ and a supermartingale in general. We know (Revuz and Yor [1999, Chapter X, Section 2]) that any continuous additive functional A_t of linear Brownian Motion can be written as

$$A_t = f(W_t) - f(W_0) - \int_0^t f'_-(W_s)\mathrm{d}W_s \tag{3.18}$$

for some convex function f. Then we must have that for any s, t,

$$\mathbb{E}\left[A_t - A_s|\mathcal{F}_t\right] = \mathbb{E}\left[f(W_t) - f(W_s)|\mathcal{F}_t\right].$$

We therefore choose $Z(x) = f(x)$ to give the result. □

We now return to proving Theorem 3.5 by choosing the function H.

Proof of Theorem 3.5. Our choice of H should be to give $F = G+H$ on the boundaries, and $F \leq G+H$ in general. We have

$$G(x,t) + Z(x) = G^*(x,t) = -\int_t^{r(x)} M(x,s)\mathrm{d}s,$$

so for any x, t

$$\begin{aligned} t < K(x) \quad &\implies \quad F_t(x,t) = -\frac{\beta^2}{2}h(x,t) \leq M(x,t) = G^*_t(x,t), \\ t > K(x) \quad &\implies \quad F_t(x,t) = 0 \geq M(x,t) = G^*_t(x,t). \end{aligned}$$

From these derivatives we can see that if $G(x, l(x)) + H(x) \geq F(x, l(x))$ and $G(x, r(x))+$

$H(x) \geq F(x, r(x))$ (where $l(x), r(x)$ are possibly $0, \infty$ respectively), then $G(x,t) + H(x) \geq F(x,t)$ everywhere, as required.

Let $H(x) = Z(x) + (\Gamma(x))_+$, so $G(x,t) + H(x) = G^*(x,t) + (\Gamma(x))_+$. This is a pathwise superhedging strategy since

$$\Gamma(x) > 0 \implies \begin{cases} G(x, l(x)) + H(x) = F(x, l(x)) \\ G(x, r(x)) + H(x) = \Gamma(x) > F(x, r(x)), \end{cases}$$
$$\Gamma(x) < 0 \implies \begin{cases} G(x, l(x)) + H(x) = F(x, l(x)) - \Gamma(x) > F(x, l(x)) \\ G(x, r(x)) + H(x) = F(x, r(x)), \end{cases}$$
$$\Gamma(x) = 0 \implies \begin{cases} G(x, l(x)) + H(x) = F(x, l(x)) \\ G(x, r(x)) + H(x) = F(x, r(x)). \end{cases}$$

For $x \in \operatorname{supp}(\mu_r)$ we require $G(x, r(x)) + H(x) = F(x, r(x))$, which holds by the above when $\Gamma(x) \leq 0$. Similarly, for $x \in \operatorname{supp}(\mu_l)$ we have $G(x, l(x)) + H(x) = F(x, l(x))$ when $\Gamma(x) \geq 0$. Also note that for $x \notin \operatorname{supp}(\mu_l) \cup \operatorname{supp}(\mu_r)$ we can choose any $H(x)$ that gives the superhedging property.

We now have the desired properties for G and H and prove our theorem as follows. Let τ' be any other stopping time that embeds μ. Then,

$$\begin{aligned} \mathbb{E}[F(W_{\tau_\mathcal{D}}, \tau_\mathcal{D})] &= \mathbb{E}[G(W_{\tau_\mathcal{D}}, \tau_\mathcal{D})] + \mathbb{E}[H(W_{\tau_\mathcal{D}})] \\ &= G(W_0, 0) + \int_\mathbb{R} H(x)\mu(\mathrm{d}x) \\ &\geq \mathbb{E}[G(W_{\tau'}, \tau')] + \int_\mathbb{R} H(x)\mu(\mathrm{d}x) \\ &\geq \mathbb{E}[F(W_{\tau'}, \tau')]. \end{aligned}$$

The first equality follows from our assumption (Γ), so, as we have shown above, our processes $G(W_t, t) + H(W_t)$ and $F(W_t, t)$ agree on the boundary. Also note that $\mathbb{E}\left[H(W_{\tau_\mathcal{D}})\right] < \infty$ since $A_{\tau_\mathcal{D}}$ is integrable. In the second line we use the martingale property of $G(W_t, t)$ in $\mathcal{D}$ and rewrite the H term as an integral to make it clear that this term does not change, since both stopping times embed μ. The inequality then follows since $G(W_t, t)$ is a supermartingale up to H_{x^*} and we know that for any embedding τ' of μ we have that $W_{\tau'} \leq x^*$. The final inequality is true since we have shown above that $G + H \geq F$ everywhere. $\square$

Remark 3.7. The case for geometric Brownian motion, under Assumption 2 is similar, noting that the measure associated with a continuous additive functional of a geometric Brownian motion is a Radon measure, and therefore we again have the representation (3.18) (see Revuz and Yor [1999, Chapter X, Section 2]).

Remark 3.8. For sufficiently smooth curves l and r, then we may find that G is differentiable, in which case the above proof can be simplified through the use of Itô's lemma to show for example that G^* is a submartingale, or that A_t depends only on W_t.

3.5 Discretisation and the Necessity of (Γ)

3.5.1 Discretisation

Our aim now is to show the converse of Theorem 3.5, that is if we have a K-cave barrier that does not satisfy (Γ), then it does not give the optimal embedding. To do this we show that the functions G, H we have chosen are the correct choice of the functions in our 'dual' problem of finding the cheapest superhedging portfolio. We have proposed one feasible superhedging portfolio, and this portfolio gives the sufficient condition (Γ), but other feasible dual formulations could give different conditions, so we show that our condition is also necessary. To show this we require some form of strong duality result, which furthermore gives the form of the dual optimisers. To the best of our knowledge these results are not available in our current setup, but we can discretise our problem and then use the results of Chapter 2.

We work under Assumption 1, with the added assumptions of Chapter 2, so suppose μ is bounded, with $x^* := \inf\{x > 0 : \mu((x,\infty)) = 0\}$ and $x_* := \sup\{x < 0 : \mu((-\infty,x)) = 0\}$. We work on the grid $\left(x_j^N, t_n^N\right) = \left(\frac{j}{\sqrt{N}}, \frac{n}{N}\right)$ for $j \in \{\lfloor x_*\sqrt{N}\rfloor, \lfloor x_*\sqrt{N}\rfloor + 1, \ldots, \lfloor x^*\sqrt{N}\rfloor\} =: \mathcal{J}$ and $n \geq 0$. Let $j_0^N := \lfloor x_*\sqrt{N}\rfloor$, $j_1^N := \lfloor x_*\sqrt{N}\rfloor + 1, \ldots, j_L^N := \lfloor x^*\sqrt{N}\rfloor$, where $L \sim \sqrt{N}$, so $\mathcal{J} = \{j_0^N, j_1^N, \ldots, j_L^N\}$. We also define $\mathcal{J}' = \{j_1^N, \ldots, j_{L-1}^N\}$, and $\mathcal{J}'' = \{j_2^N, \ldots, j_{L-2}^N\}$. For each N we choose $j^{*,N} \in \mathcal{J}$ so that $x_{j^*}^N = \frac{j^{*,N}}{\sqrt{N}} \to 0$ as $N \to \infty$. If Y^N is the simple symmetric random walk on this grid, started at $x_{j^*}^N$, then by Donsker's Theorem, $Y_{\lfloor Nt\rfloor}^N$ converges in distribution to a Brownian motion started at 0. In the case of geometric Brownian motion we take $x_j^N = \mathrm{e}^{\frac{j}{\sqrt{N}}}$.

We also need a discretised version of our payoff F, say $\bar{F}^N$, chosen so $\bar{F}^N(\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor) \to F(x,t)$ everywhere. Under Assumption 1 our continuous-time payoff function is $F(x,t) = \left(\mathrm{e}^{\beta x}\mathrm{e}^{-\frac{\beta^2}{2}t} - k\right)_+ = (h(x,t) - k)_+$, where $h(X_t,t)$ is a martingale, and we write the discretised version with a similar martingale term. We have

$$\begin{aligned}\mathbb{E}\left[\exp(\beta Y_{n+1}^N)|Y_n^N\right] &= \exp(\beta Y_n^N)\left(\frac{1}{2}\exp\left(\frac{\beta}{\sqrt{N}}\right)+\frac{1}{2}\exp\left(-\frac{\beta}{\sqrt{N}}\right)\right)\\ &= \exp(\beta Y_n^N)\cosh\left(\frac{\beta}{\sqrt{N}}\right),\end{aligned}$$

and so $\bar{F}^N(j,t) = \bar{F}^N_{j,t} := \left(\mathrm{e}^{\beta x_j^N}\left(\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right)^{-t} - k\right)_+$ has the same form as before. Note now that $\bar{F}^N_{j,n} \approx F(x_j^N, n\Delta t) = F(\frac{j}{\sqrt{N}}, \frac{n}{N})$, or $F(x,t) \approx \bar{F}^N_{\lfloor x\sqrt{N}\rfloor,\lfloor tN\rfloor}$, since $\left(\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right)^{-Nt} \to \mathrm{e}^{-\frac{\beta^2}{2}t}$, as $N \to \infty$. In the case of Assumption 2 the arguments are the same.

We can now consider the problems $\mathcal{P}^N$ and $\mathcal{D}^N$ as introduced in Chapter 2 with the above $\bar{F}^N$. In Chapter 2 we show that we have strong duality in the sense that the optimal values of these problems are equal, and both values are obtained by some optimal p^*, ν^*, η^*. The original primal-dual pair optimises over $(p_{j,t}) \in l^1(\lambda) := \left\{(x_{j,t}) : \sum_{j,t}|x_{j,t}|\lambda^t < \infty\right\}$ and $(\nu_j, \eta_{j,t}) \in \mathbb{R}^{L+1}\times l^\infty(\lambda^{-1})$, where $l^\infty(\lambda^{-1}) := \left\{(y_{j,t}) : \sup_{j,t}|y_{j,t}|\lambda^{-t} < \infty\right\}$ and $\lambda > 1$ is a constant. The duality result Theorem 2.5 gives dual optimisers $(\nu_j^*, \eta_{j,t}^*) \in \mathbb{R}^{L+1}\times l^\infty(\lambda^{-1})$, however for the primal optimisers we can only argue that there is an optimal sequence $(p_{j,t}^*) \in l^1$, not $l^1(\lambda)$. Note further that since $F(W_t,t)$ is a submartingale, we can apply the proof of Lemma 2.8 to $\bar{F}^N$ to show that the $p_{j,t}^*$ give a stopping rule σ^N which embeds the correct discretised distribution, so $Y^N_{\sigma^N} \sim \mu^N$.

To ensure that the dual variables are in the true dual space of the primal variables, we require $(\nu_j^*, \eta_{j,t}^*) \in \mathbb{R}^{L+1}\times l^\infty$. Note that for large T (such that $\bar{F}^N_{j,t} = 0$ for all $t \geq T$), $\eta_{j,t}^T = \eta_{j,t}^*\mathbf{1}\{t < T\}$ gives a feasible sequence $(\eta_{j,t}^T) \in l^\infty$, and this sequence also gives the same value of the objective function. We can therefore, without loss of generality, restrict our dual problem to $\mathbb{R}^{L+1}\times l^\infty$.

With our setup complete, we can now adapt Theorem 2.10 to prove a discrete version of Theorem 3.2.

Theorem 3.9. *The optimal solution of the primal problem $\mathcal{P}^N$, where $\bar{F}^N_{j,t}$ is our discretised LETF function, is given by a sequence $(p_{j,t}^*)$ which gives a stopping region for a random walk with the K-cave barrier-like property*

$$\text{if } q_{i,t}^* > 0 \text{ for some } (i,t) \text{ where } t < K(x_i^N), \text{ then } p_{i,s}^* = 0\ \forall s < t, \tag{3.19}$$

$$\text{if } q_{i,t}^* > 0 \text{ for some } (i,t) \text{ where } t > K(x_i^N), \text{ then } p_{i,s}^* = 0\ \forall s > t. \tag{3.20}$$

Proof. First consider the inverse-barrier to the left of the curve K. To show (3.19), suppose we have a feasible solution with $q_{i,t} > 0$ and $p_{i,s} > 0$ for some i and $s < t < K(x_i^N)$. We take some $0 < \varepsilon < \min\{\frac{1}{2}q_{i,t}, p_{i,s}\}$ and show that we can improve our objective function by transferring ε of the mass that currently leaves (i,s) onto (i,t). We use the $\tilde{p}$, $\tilde{q}$, $\bar{p}$, $\bar{q}$ defined in Theorem 2.10, and the feasibility of these new probabilities is exactly as in Lemma 2.11.

Now, $\bar{F}_{j,t} = \left(\bar{h}_{j,t} - k\right)_+$ where $\bar{h}_{Y_t,t}$ is a martingale, so $\sum_{r>s,j} \bar{h}_{j,r}\tilde{q}_{j,r} = \varepsilon\bar{h}_{i,s}$. Let $K_j = K(x_j^N)$, then for any j we have $\{r > s\} = \{s < r \le K_j - (t-s)\} \cup \{K_j - (t-s) < r \le K_j\} \cup \{r > K_j\}$. Fix some j such that $s < K_j$, then we have

$$
\begin{aligned}
&\bar{F}_{j,r+t-s} - \bar{F}_{j,r} = \bar{h}_{j,r+t-s} - \bar{h}_{j,r}, && \text{in } \{s < r \le K_j - (t-s)\},\\
&\bar{F}_{j,r+t-s} - \bar{F}_{j,r} = k - \bar{h}_{j,r} \ge \bar{h}_{j,r+t-s} - \bar{h}_{j,r}, && \text{in } \{K_j - (t-s) < r \le K_j\},\\
&\bar{F}_{j,r+t-s} - \bar{F}_{j,r} = 0 \ge \bar{h}_{j,r+t-s} - \bar{h}_{j,r}, && \text{in } \{r > K_j\}.
\end{aligned}
$$

Combining these, we see that

$$
\begin{aligned}
\sum_{j,r} \bar{F}_{j,r}\bar{q}_{j,r} &= \sum_{j,r} \bar{F}_{j,r}q_{j,r} + \varepsilon(\bar{F}_{i,s} - \bar{F}_{i,t}) - \sum_{r>s,j} \bar{F}_{j,r}\tilde{q}_{j,r} + \sum_{r>t,j} \bar{F}_{j,r}\tilde{q}_{j,r-(t-s)}\\
&= \sum_{j,r} \bar{F}_{j,r}q_{j,r} + \varepsilon(\bar{F}_{i,s} - \bar{F}_{i,t}) + \sum_{r>s,j} \tilde{q}_{j,r}\left(\bar{F}_{j,r+t-s} - \bar{F}_{j,r}\right)\\
&\ge \sum_{j,r} \bar{F}_{j,r}q_{j,r} + \varepsilon(\bar{h}_{i,s} - \bar{h}_{i,t}) + \sum_{r>s,j} \tilde{q}_{j,r}\left(\bar{h}_{j,r+t-s} - \bar{h}_{j,r}\right)\\
&= \sum_{j,r} \bar{F}_{j,r}q_{j,r} + \sum_{r>s,j} \tilde{q}_{j,r}(\bar{h}_{j,r} - \bar{h}_{j,r+t-s}) + \sum_{r>s,j} \tilde{q}_{j,r}\left(\bar{h}_{j,r+t-s} - \bar{h}_{j,r}\right)\\
&= \sum_{j,r} \bar{F}_{j,r}q_{j,r}.
\end{aligned}
$$

The right hand barrier (3.20) is similar, and we use $\hat{p}, \hat{q}$ defined in Theorem 2.10. Now we have that $\bar{F}_{j,r} = 0$ for $r > K(x_j^N)$ and this simplifies our argument:

$$
\begin{aligned}
\sum_{j,r} \bar{F}_{j,r}\hat{q}_{j,r} &= \sum_{j,r} \bar{F}_{j,r}q_{j,r} - \sum_{r>s,j} \bar{F}_{j,r}\tilde{q}_{j,r} + \sum_{r>t,j} \bar{F}_{j,r}\tilde{q}_{j,r+s-t}\\
&= \sum_{j,r} \bar{F}_{j,r}q_{j,r} + \sum_{r>s,j} (\bar{F}_{j,r-(s-t)} - \bar{F}_{j,r})\tilde{q}_{j,r}\\
&\ge \sum_{j,r} \bar{F}_{j,r}q_{j,r},
\end{aligned}
$$

since $\bar{F}_{j,r}$ is decreasing in r.

We have improved the value of our objective function and therefore any solution without this K-cave property is suboptimal. Since we know that optimisers exist, they must have this property. $\square$

From Lemma 2.6 and Theorem 3.9 we know that an optimal solution exists for each $\mathcal{P}^N$ and this is a sequence $(p^{*,N})$ that corresponds to a stopped random walk that is stopped by some almost-deterministic stopping region $\hat{\mathcal{B}}^N$ that takes the form of a K-cave barrier. The region $\hat{\mathcal{B}}^N$ is determined by points $\bar{l}_j^N$ and $\bar{r}_j^N$, defined as the largest time $\bar{l}_j^N < K(x_j^N)$ such that $p_{j,t}^{*,N} = 0\ \forall t \leq \bar{l}_j^N$, and similarly the smallest time $\bar{r}_j^N > K(x_j^N)$ such that $p_{j,t}^{*,N} = 0\ \forall t \geq \bar{r}_j^N$. Note that for each j we either have $q_{j,\bar{r}_j^N}^{*,N} > 0$, or $q_{j,s}^{*,N} = 0\ \forall s > K(x_j^N)$, and similarly for $\bar{l}_j^N$. These barriers have equivalent stopping regions, $\mathcal{B}^N$, for a Brownian motion, and Lemma 2.19 says that these barriers converge to a continuous time K-cave barrier $\mathcal{B}^\infty$ which embeds μ into a Brownian motion. From Lemma 2.20 we know that the corresponding stopping time is indeed a maximiser of (OptSEP), and in fact that the stopped random walks converge to the stopped Brownian motion. In other words, if P^N is the optimal value of $\mathcal{P}^N$, then $\mathrm{P}^N \to \sup_{\tau, W_\tau \sim \mu} \mathbb{E}\left[F(W_\tau, \tau)\right]$, and our discrete barriers converge exactly to an optimal stopping region for (OptSEP). This approach therefore reproves Theorem 3.2. Furthermore, we can now look at the convergence of the dual optimisers η^*, ν^*.

3.5.2 Dual Convergence

We know by strong duality that an optimal solution to the linear programming problem is given by the p, q, ν, η that are $\mathcal{P}^N$-feasible and $\mathcal{D}^N$-feasible, and for which the complementary slackness conditions hold. In Theorem 3.5 we show that if τ is such that certain properties of G, H hold, then we have optimality, and as shown in Section 2.3.2, the complementary slackness conditions here have obvious connections to these properties. Once we have convergence it will guarantee the correct choice of our functions G, H and therefore show that (Γ) is both a necessary and sufficient condition for optimality.

Let τ be an optimiser of (OptSEP) of the form of a hitting time of a K-cave barrier, which we know exists by Theorem 3.2, or from the limiting arguments of the previous section. Recall that $G(x,t) = -\int_t^{r(x)} M(x,s)\mathrm{d}s - Z(x)$, where $M(x,t) = \mathbb{E}^{x,t}\left[\partial_t F(W_\tau, \tau)\right]$, and now we show that our dual optimisers $\eta^{*,N}$ take a similar form. Fix N and let $\mathcal{D} := \left\{(j,t) : p_{j,t}^{*,N} > 0\right\}$. For presentation purposes we will drop the dependence on N in much of what follows, so let $\bar{\tau}$ be the stopping law of our random walk Y in the N-grid given by the $p^*_{j,t}$ (or $\bar{\tau}^{j,t}$ if Y starts at (j,t)). We will also write $\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} := \bar{F}^N(\sqrt{N}Y_{\bar{\tau}},\bar{\tau})$. Then for $(j,t) \in \mathcal{D}$, since we have a positive probability of leaving (j,t), we have $q^*_{Y_{\bar{\tau}},\bar{\tau}} > 0$ almost surely, and so by (2.10), $\eta^*_{Y_{\bar{\tau}},\bar{\tau}} = 0$. Since we have the interpretation that η^* represents $G+H-F$, write $\tilde{\eta}^* = \eta^* + \bar{F}^N$. From (2.9) we deduce that

$$\tilde{\eta}^*_{j,t} = \mathbb{E}^{j,t}\left[\eta^*_{Y_{\bar{\tau}},\bar{\tau}} + \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \sum_{s=t}^{\bar{\tau}-1} \nu^*_{Y_s}\right] = \mathbb{E}^{j,t}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \sum_{s=t}^{\bar{\tau}-1} \nu^*_{Y_s}\right].$$

Now define a new stopping time as $\left(\bar{\tau}^{-1}\right)^{j,t-1} = \inf\left\{n \geq t-1 : \left(Y^{j,t-1}_n, n+1\right) \notin \mathcal{D}\right\}$. By the strong Markov property we see that $\left(\bar{\tau}^{-1}\right)^{j,t-1} = \bar{\tau}^{j,t} - 1 \geq t-1$, and $Y^{j,t-1}_{\bar{\tau}^{-1}} = Y^{j,t}_{\bar{\tau}}$. Now,

$$\begin{aligned}
\tilde{\eta}^*_{j,t-1} &\geq \mathbb{E}^{j,t-1}\left[\eta^*_{Y_{\bar{\tau}^{-1}},\bar{\tau}^{-1}} + \bar{F}^N_{Y_{\bar{\tau}^{-1}},\bar{\tau}^{-1}} - \sum_{s=t}^{\bar{\tau}^{-1}-1} \nu^*_{Y_s}\right] && \text{by (2.4)} \\
&\geq \mathbb{E}^{j,t-1}\left[\bar{F}^N_{Y_{\bar{\tau}^{-1}},\bar{\tau}^{-1}} - \sum_{s=t}^{\bar{\tau}^{-1}-1} \nu^*_{Y_s}\right] && \text{by (2.3)} \\
&= \mathbb{E}^{j,t}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}-1} - \sum_{s=t}^{\bar{\tau}-2} \nu^*_{Y_s}\right].
\end{aligned}$$

We then have

$$\tilde{\eta}^*_{j,t} - \tilde{\eta}^*_{j,t-1} \leq \mathbb{E}^{j,t}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}-1} - \nu^*_{Y_{\bar{\tau}-1}}\right] \leq \mathbb{E}^{j,t}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}-1}\right].$$

In a very similar fashion we can find a lower bound, giving us

$$\mathbb{E}^{j,t-1}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}+1} - \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}}\right] \leq \tilde{\eta}^*_{j,t} - \tilde{\eta}^*_{j,t-1} \leq \mathbb{E}^{j,t}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}-1}\right] \leq 0.$$

From the form of $\bar{F}^N$ under Assumption 1 (geometric Brownian motion is similar) we deduce that for $t < K(x^N_j)$, $\bar{F}^N_{j,t} - \bar{F}^N_{j,t-1} = \mathrm{e}^{\beta x^N_j}\left(\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right)^{-t}\left(1 - \cosh\left(\frac{\beta}{\sqrt{N}}\right)\right)$. In particular, we have that

$$N\left(\bar{F}^N_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} - \bar{F}^N_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor-1}\right) \to \partial_t F(x,t), \quad \forall (x,t).$$

In Lemmas 2.19 and 2.20 we show that $\left|\left(\frac{\bar{\tau}^N}{N}, Y_{\bar{\tau}^N}\right) - \left(\tau^N, W_{\tau^N}\right)\right| \xrightarrow{d} 0$, and $\left(\tau^N, W_{\tau^N}\right) \xrightarrow{\mathbb{P}} (\tau, W_\tau)$ as $N \to \infty$, where τ is an optimiser of (OptSEP) and τ^N is the Brownian hitting time of the K-cave barrier $\mathcal{B}^N$. Both $\bar{F}^N$ and F are bounded in our domain and Lipschitz continuous in time, so for $t < K(x)$ we have

$$N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}-1}\right] \to \mathbb{E}^{x,t}\left[\partial_t F(W_\tau,\tau)\right], \quad \text{as } N \to \infty.$$

We can now find the limit of our dual optimisers $\tilde{\eta}^*$.

For any x, let $\bar{r}^N_x$ denote the left-most point of the right-hand barrier at level $\lfloor\sqrt{N}x\rfloor$ of $\hat{\mathcal{B}}^N$. Then $r(x) := \lim_{N\to\infty}\frac{\bar{r}^N_x}{N} \in [K(x),\infty]$ is the left-most point of the right hand boundary at x of the limit barrier $\mathcal{B}^\infty$.

Lemma 3.10. *For any (x,t) in our domain,*

$$\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} - \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}^N_x} \to \int_{r(x)}^t \mathbb{E}^{x,s}\left[\partial_t F(W_\tau,\tau)\right]\mathrm{d}s \quad \textit{as } N \to \infty.$$

Proof. Suppose first $r(x) < \infty$. If $t > r(x)$ then $\exists N_0$ such that $N \geq N_0 \implies Nt > \bar{r}^N_x$ and then $\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} = 0$ by (2.10) and we are done. Suppose $t < r(x)$, then for large N we know by the above that

$$\begin{aligned}
-\sum_{s=\lfloor Nt\rfloor+1}^{\bar{r}^N_x}\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,s-1}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}+1} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}}\right] \geq\ & \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} - \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}^N_x} \\
&\geq -\sum_{s=\lfloor Nt\rfloor+1}^{\bar{r}^N_x}\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,s}\left[\bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}} - \bar{F}^N_{Y_{\bar{\tau}},\bar{\tau}-1}\right].
\end{aligned}$$

We look at the convergence of the right-hand side and argue that the other inequality is similar. First note that when $\bar{r}^N_x < \infty$, we know $q_{j,\bar{r}^N_x} > 0$, and so by our complementary slackness conditions, $\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}^N_x} = \bar{F}^N_{\lfloor\sqrt{N}x\rfloor,\bar{r}^N_x} = 0$, since $\eta^* = 0$ in the stopping region.

Now,

$$\begin{aligned}
\sum_{s=\lfloor Nt\rfloor+1}^{\bar{r}_x^N} \mathbb{E}^{\lfloor\sqrt{N}x\rfloor,s}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right] &= \sum_{s=\frac{\lfloor Nt\rfloor+1}{N}}^{\frac{\bar{r}_x^N}{N}} \mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right] \\
&= \sum_{s=\frac{\lfloor Nt\rfloor+1}{N}}^{\frac{\bar{r}_x^N}{N}} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\frac{1}{N} \\
&= \int_{\frac{(\lfloor Nt\rfloor+1)}{N}}^{\frac{\bar{r}_x^N}{N}} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\mathrm{d}s \\
&= \int_t^{r(x)} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\mathrm{d}s \\
&\qquad + \int_{\frac{(\lfloor Nt\rfloor+1)}{N}}^{t} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\mathrm{d}s \\
&\qquad\quad + \int_{r(x)}^{\frac{\bar{r}_x^N}{N}} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\mathrm{d}s.
\end{aligned}$$

Since we are working in $[x_*, x^*]$ we see that the integrand above is non-positive and bounded below, and also

$$\begin{aligned}
N\left(\bar{F}^N_{j,t} - \bar{F}^N_{j,t-1}\right) &\geq N\mathrm{e}^{\beta x_j^N}\left(\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right)^{-t}\left(1-\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right) \\
&\geq N\mathrm{e}^{\beta x^*}\left(1-\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right) \\
&\to -\frac{\beta^2}{2}\mathrm{e}^{\beta x^*},
\end{aligned}$$

as $N \to \infty$. Then the two remainder integral terms vanish, since

$$\begin{aligned}
\left|\int_{\frac{\lfloor Nt\rfloor+1}{N}}^{t} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\mathrm{d}s\right| &\leq \left(t - \frac{\lfloor Nt\rfloor+1}{N}\right)N\mathrm{e}^{\beta x^*}\left(1-\cosh\left(\frac{\beta}{\sqrt{N}}\right)\right) \\
&\to 0, \quad \text{as } N\to\infty.
\end{aligned}$$

and similarly for the other integral since $\frac{\bar{r}_x^N}{N} - r(x) \to 0$. Finally, by the Dominated Convergence Theorem,

$$-\int_t^{r(x)} N\mathbb{E}^{\lfloor\sqrt{N}x\rfloor,\lfloor Ns\rfloor}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}-1}\right]\mathrm{d}s \to -\int_t^{r(x)} \mathbb{E}^{x,s}\left[\partial_t F(W_\tau,\tau)\right]\mathrm{d}s.$$

If $r(x) = \infty$, then the integral on the right hand-side above is still finite since we are working on a bounded domain and $F = 0$ for large t. In this case the same argument holds once we observe that only finitely many terms in each of our sums can be non-zero.

The other inequality is similar, and shows

$$-\sum_{s=\lfloor Nt\rfloor+1}^{\bar{r}_x^N} \mathbb{E}^{\lfloor\sqrt{N}x\rfloor,s-1}\left[\bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}+1} - \bar{F}^N_{Y_{\tilde{\tau}},\tilde{\tau}}\right] \to -\int_t^{r(x)} \mathbb{E}^{x,s}\left[\partial_t F(W_\tau,\tau)\right] \mathrm{d}s.$$

Recall that since τ is an optimiser, it embeds no mass along the curve K, and F is differentiable away from K, so without loss of generality we can write $\partial_t F(x,t)$. ☐

We can now prove that our discrete dual optimisers converge to exactly the dual solution we gave earlier, and that we therefore have strong duality in the continuous time problem, but first we look at the effect of the $\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N}$ term in the above. Recall that we define $\Gamma(x) := \int_{l(x)}^{r(x)} M(x,s)\mathrm{d}s + F(x,l(x))$, so by Lemma 3.10,

$$\begin{aligned}\Gamma(x) &= \lim_{N\to\infty}\left(-\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} + \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} + \bar{F}^N_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N}\right)\\ &= \lim_{N\to\infty}\left(\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} - \eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N}\right).\end{aligned}$$

Also recall that we define μ_l to be the the distribution of the mass embedded along $l(x)$, i.e. the distribution of the stopped Brownian motion when it hits l before r, and μ_r similarly. Then, for $x \in \operatorname{supp}(\mu_r)$, $\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} \geq 0$ and $\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} = 0$ by (2.10), so

$$\Gamma(x) = \lim_{N\to\infty}\left(\bar{F}^N_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} - \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N}\right) = -\lim_{N\to\infty}\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} \leq 0,$$

and therefore

$$\lim_{N\to\infty}\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} = 0 = (\Gamma(x))_+.$$

For $x \in \operatorname{supp}(\mu_l)$, $\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} \geq 0$ and $\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} = 0$ by (2.10), so

$$\Gamma(x) = \lim_{N\to\infty}\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} \geq 0 \implies \lim_{N\to\infty}\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} = (\Gamma(x))_+.$$

In particular we have proven the following.

Lemma 3.11. *For any $x \in \operatorname{supp}(\mu_l)\cup\operatorname{supp}(\mu_r)$, $\lim_{N\to\infty}\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} = (\Gamma(x))_+$. Furthermore, in the limiting K-cave barrier $\mathcal{B}^\infty$, (Γ) holds.*

We have shown that the condition (Γ) holds in our limiting stopping region, and all that remains to show is that with our functions G and H from Theorem 3.5 there is no duality gap.

Theorem 3.12. *With $G(x,t)$, $H(x)$ defined as in Theorem 3.5,*

$$\sup_{\tau, W_\tau \sim \mu} \mathbb{E}\left[F(W_\tau, \tau)\right] = \mathbb{E}\left[G(W_\tau, \tau) + H(W_\tau)\right].$$

Proof. By Lemma 3.10 and Lemma 3.11, $\lim_{N\to\infty} \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor} = G^*(x,t) + (\Gamma(x))_+ = G(x,t) + H(x)$ for G, H as in Theorem 3.5 (for $x \notin \text{supp}(\mu_l) \cup \text{supp}(\mu_r)$ we can ensure this by our freedom of choice of $H(x)$). We can write $\tilde{\eta}^*_{j,t} = \tilde{\eta}^{*,G}_{j,t} + \tilde{\eta}^{*,H}_{j,t}$ such that $\tilde{\eta}^{*,G}_{j,t}$ is a martingale and $\lim_{N\to\infty} \tilde{\eta}^{*,G}_{\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor} = G(x,t)$ for any (x,t). Then clearly $\lim_{N\to\infty} \tilde{\eta}^{*,H}_{\lfloor\sqrt{N}x\rfloor, \lfloor Nt\rfloor} = H(x)$ for any (x,t), and so $\tilde{\eta}^{*,H}_{j,t} = \tilde{\eta}^{*,H}_j$ is independent of t.

When N is sufficiently large, for every $j \neq j_0^N, j_L^N$ there is some t such that $p_{j,t} > 0$, so from (2.9) we have

$$\nu_j = \frac{1}{2}\left(\tilde{\eta}^*_{j+1,t+1} + \tilde{\eta}^*_{j-1,t+1}\right) - \tilde{\eta}^*_{j,t} = \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j+1} + \tilde{\eta}^{*,H}_{j+1}\right) - \tilde{\eta}^{*,H}_j.$$

From the ideas in Section 2.3.2 we suspect that $N\nu_{\lfloor\sqrt{N}x\rfloor} \to \dfrac{1}{2}H''(x)$ as $N \to \infty$. Since we cannot argue the convergence of derivatives, the corresponding summation is

$$\begin{aligned}
\sum_{m=1}^{i}\sum_{k=1}^{m} \nu_{j_k} &= \sum_{m=1}^{i}\sum_{k=1}^{m} \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_{k+1}} + \tilde{\eta}^{*,H}_{j_{k-1}}\right) - \tilde{\eta}^{*,H}_{j_k} \\
&= \sum_{m=1}^{i}\left(\frac{1}{2}\sum_{k=2}^{m+1}\tilde{\eta}^{*,H}_{j_k} + \frac{1}{2}\sum_{k=0}^{m-1}\tilde{\eta}^{*,H}_{j_k} - \frac{1}{2}\sum_{k=1}^{m}\tilde{\eta}^{*,H}_{j_k}\right) \\
&= \frac{1}{2}\sum_{m=1}^{i}\left(\left(\tilde{\eta}^{*,H}_{j_{m+1}} - \tilde{\eta}^{*,H}_{j_m}\right) - \left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0}\right)\right) \\
&= \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1}\right) - \frac{1}{2}i\left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0}\right),
\end{aligned}$$

and in particular,

$$\lim_{N\to\infty}\sum_{m=1}^{\lfloor\sqrt{N}x\rfloor}\sum_{j=j_0}^{m}\nu_j = \frac{1}{2}\left(H(x) - H(x_*)\right) - \frac{1}{2}\lim_{N\to\infty}\lfloor\sqrt{N}x\rfloor\left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0}\right).$$

Our aim is to rewrite $\sum_{j\in\mathcal{J}'}\nu_j U_j$ to incorporate this double sum by an integration by parts type argument and work instead with the derivatives of U. Let $V_{j_i} = U_{j_{i+1}} - U_{j_i}$, $W_{j_i} = V_{j_{i+1}} - V_{j_i}$ for $i = 0, \ldots, L-1$, $V_{j_L} = W_{j_L} = 0$, and $\nu_0 = 0$. Then, noting that

$U_{j_L} = 0,$

$$\begin{aligned}
\sum_{j\in\mathcal{J}'} \nu_j U_j &= \sum_{i=1}^{L-1} \left(\sum_{k=0}^{i} \nu_{j_k} - \sum_{k=0}^{i-1} \nu_{j_k} \right) U_{j_i} \\
&= -\sum_{i=1}^{L-1} \left(\sum_{k=0}^{i} \nu_{j_k} \right) V_{j_i} + \left(\sum_{k=1}^{L-1} \nu_{j_k} \right) U_{j_L} \\
&= -\sum_{i=1}^{L-1} \left(\sum_{m=0}^{i} \sum_{k=0}^{m} \nu_{j_k} - \sum_{m=0}^{i-1} \sum_{k=0}^{m} \nu_{j_k} \right) V_{j_i} \\
&= \sum_{i=1}^{L-2} \left(\sum_{m=1}^{i} \sum_{k=1}^{m} \nu_{j_k} \right) W_{j_i} - \left(\sum_{i=1}^{L-1} \sum_{k=1}^{i} \nu_{j_k} \right) V_{j_{L-1}}.
\end{aligned}$$

Substituting in our expression for the double summation of ν, the first term becomes

$$\begin{aligned}
\sum_{i=1}^{L-2} \left(\sum_{m=1}^{i} \sum_{k=1}^{m} \nu_{j_k} \right) W_{j_i} &= \sum_{i=1}^{L-2} \left(\frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1} \right) - \frac{1}{2} i \left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0} \right) \right) W_{j_i} \\
&= \sum_{i=1}^{L-2} \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1} \right) W_{j_i} \\
&\qquad - \sum_{i=1}^{L-2} \frac{1}{2} i \left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0} \right) \left(V_{j_{i+1}} - V_{j_i} \right) \\
&= \sum_{i=1}^{L-2} \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1} \right) W_{j_i} \\
&\qquad - \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0} \right) \left((L-1) V_{j_{L-1}} - \sum_{i=1}^{L-1} V_{j_i} \right) \\
&= \sum_{i=1}^{L-2} \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1} \right) W_{j_i} \\
&\qquad - \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0} \right) \left(V_{j_0} + (L-1) V_{j_{L-1}} \right).
\end{aligned}$$

For the second term,

$$\left(\sum_{i=1}^{L-1} \sum_{k=1}^{i} \nu_{j_k} \right) V_{j_{L-1}} = \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_L} - \tilde{\eta}^{*,H}_{j_1} \right) V_{j_{L-1}} - \frac{1}{2} \left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0} \right) (L-1) V_{j_{L-1}},$$

and so

$$\sum_{j\in\mathcal{J}'}\nu_j U_j = \sum_{i=1}^{L-2}\frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1}\right)W_{j_i} - \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0}\right)V_{j_0} - \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_L} - \tilde{\eta}^{*,H}_{j_1}\right)V_{j_{L-1}}.$$

To work with the derivatives of the potential, we now approximate it by smooth functions. From our choice of U^N, for each x we know $\frac{1}{\sqrt{N}}U^N(\lfloor\sqrt{N}x\rfloor) \to U_{\delta_0}(x) - U_\mu(x)$, the difference in the potential functions of the distributions δ_0 and μ. These functions are continuous and concave so by the Stone-Weierstrass theorem there exists a decreasing sequence of functions, $(\tilde{U}^n)_n$, in C^∞ converging uniformly to $U_{\delta_0} - U_\mu$ with $\tilde{U}^n(x_*) = \tilde{U}^n(x^*) = 0$ for all n. For a given n we can find discrete approximations $\tilde{U}^{n,N}$ of $\tilde{U}^n$ (and the associated $\tilde{V}^{n,N}$, $\tilde{W}^{n,N}$) such that $\frac{1}{\sqrt{N}}\tilde{U}^{n,N}_{\lfloor\sqrt{N}x\rfloor} \to \tilde{U}^n(x)$, $\tilde{V}^{n,N}_{\lfloor\sqrt{N}x\rfloor} \to \frac{\mathrm{d}\tilde{U}^n}{\mathrm{d}x}(x)$, and $\sqrt{N}\tilde{W}^{n,N}_{\lfloor\sqrt{N}x\rfloor} \to \frac{\mathrm{d}^2\tilde{U}^n}{\mathrm{d}x^2}(x)$ for all x as $N\to\infty$. We can also, without loss of generality, choose $\tilde{U}^{n,N}$ such that $\tilde{U}^{n,N}_{j_0} = \tilde{U}^{n,N}_{j_L} = 0$ and $\tilde{U}^{n,N} \geq \tilde{U}^n$.

Then, by the above,

$$\begin{aligned}
\sum_{j\in\mathcal{J}'}\nu_j\tilde{U}^{n,N}_j &= \sum_{i=1}^{L-2}\frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_{i+1}} - \tilde{\eta}^{*,H}_{j_1}\right)\tilde{W}^{n,N}_{j_i} - \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0}\right)\tilde{V}^{n,N}_{j_0} \\
&\qquad - \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_L} - \tilde{\eta}^{*,H}_{j_1}\right)\tilde{V}^{n,N}_{j_{L-1}} \\
&= \int_{\frac{1}{\sqrt{N}}}^{\frac{L-2}{\sqrt{N}}}\frac{1}{2}\left(\tilde{\eta}^{*,H}_{\lfloor\sqrt{N}x\rfloor+1} - \tilde{\eta}^{*,H}_{j_1}\right)\sqrt{N}\tilde{W}^{n,N}_{\lfloor\sqrt{N}x\rfloor}\mathrm{d}x - \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_1} - \tilde{\eta}^{*,H}_{j_0}\right)\tilde{V}^{n,N}_{j_0} \\
&\qquad - \frac{1}{2}\left(\tilde{\eta}^{*,H}_{j_L} - \tilde{\eta}^{*,H}_{j_1}\right)\tilde{V}^{n,N}_{j_{L-1}} \\
&\to \int\frac{1}{2}\left(H(x) - H(x_*)\right)\frac{\mathrm{d}^2\tilde{U}^n}{\mathrm{d}x^2}(x)\mathrm{d}x - \frac{1}{2}\left(H(x^*) - H(x_*)\right)\frac{\mathrm{d}\tilde{U}^n}{\mathrm{d}x}(x^*),
\end{aligned}$$

as $N\to\infty$. Since H is convex, it is differentiable almost everywhere and has a second derivative in the sense of distributions. Using integration by parts again,

$$\begin{aligned}
\int\frac{1}{2}\left(H(x) - H(x_*)\right)\frac{\mathrm{d}^2\tilde{U}^n}{\mathrm{d}x^2}(x)\mathrm{d}x &= \int\frac{1}{2}H(x)\frac{\mathrm{d}^2\tilde{U}^n}{\mathrm{d}x^2}(x)\mathrm{d}x \\
&\qquad - \frac{1}{2}H(x_*)\left(\frac{\mathrm{d}\tilde{U}^n}{\mathrm{d}x}(x^*) - \frac{\mathrm{d}\tilde{U}^n}{\mathrm{d}x}(x_*)\right) \\
&= \int\frac{1}{2}H''(x)\tilde{U}^n(x)\mathrm{d}x + \frac{1}{2}\left(H(x^*) - H(x_*)\right)\frac{\mathrm{d}\tilde{U}^n}{\mathrm{d}x}(x^*).
\end{aligned}$$

Therefore,

$$\sum_{j\in\mathcal{J}'}\nu_j\tilde{U}_j^{n,N}\to\int\frac{1}{2}H''(x)\tilde{U}^n(x)\mathrm{d}x\quad\text{as } N\to\infty,$$

and so by monotone convergence

$$\begin{aligned}\lim_{n,N\to\infty}\sum_{j\in\mathcal{J}'}\nu_j\tilde{U}_j^{n,N}&=\int\frac{1}{2}H''(x)U(x)\mathrm{d}x\\&=\int\frac{1}{2}H''(x)\left(U_{\delta_0}(x)-U_\mu(x)\right)\mathrm{d}x\\&=\mathbb{E}\left[H(W_\tau)\right]-H(W_0).\end{aligned}$$

By our choice of approximation we know that $\lim_{N\to\infty}U^N=U\leq\tilde{U}^n=\lim_{N\to\infty}\tilde{U}^{n,N}$ for all n, and so without loss of generality we can choose $\tilde{U}^{n,N}\geq U^N$ for large n,N. Then, since $\nu_j\geq 0$ for all j, by monotone convergence it follows that

$$\left|\sum_{j\in\mathcal{J}'}\nu_j\tilde{U}_j^{n,N}-\sum_{j\in\mathcal{J}'}\nu_jU^n\right|\to 0,\quad\text{as } n,N\to\infty.$$

Finally recall that $\mathrm{D}^N=\sum_{j\in\mathcal{J}'}\nu_jU_j+\frac{1}{2}\left(\eta_{j^*+1,1}+\eta_{j^*-1,1}\right)+\frac{1}{2}\left(\bar{F}^N_{j^*+1,1}+\bar{F}^N_{j^*-1,1}\right)$, so

$$\begin{aligned}\lim_{N\to\infty}\mathrm{D}^N&=\mathbb{E}\left[H(W_\tau)\right]-H(W_0)+G^*(W_0,0)\\&=\mathbb{E}\left[H(W_\tau)\right]+G(W_0,0)\\&=\mathbb{E}\left[G(W_\tau,\tau)+H(W_\tau)\right].\end{aligned}$$

Then by the above and the results of Chapter 2,

$$\sup_{\tau,W_\tau\sim\mu}\mathbb{E}\left[F(W_\tau,\tau)\right]=\lim_{N\to\infty}\mathrm{P}^N=\lim_{N\to\infty}\mathrm{D}^N=G(W_0,0)+\mathbb{E}\left[H(W_\tau)\right].$$

$\square$

Theorem 3.13. *For a K-cave stopping time τ given by curves l, r, the condition (Γ) is necessary for optimality.*

Proof. By Theorem 3.12 our functions G and H give no duality gap. We know that

$G(x,t) + H(x) \geq F(x,t)$ everywhere, but also

$$\begin{aligned}\Gamma(x) > 0 &\implies G(x, r(x)) + H(x) > F(x, r(x)),\\ \Gamma(x) < 0 &\implies G(x, l(x)) + H(x) > F(x, l(x)),\end{aligned}$$

so if (Γ) does not hold then $\mathbb{E}[F(W_\tau, \tau)] < G(W_0, 0) + \mathbb{E}[H(W_\tau)]$, contradicting Theorem 3.12. □

3.5.3 An Additional Property of the Barrier

We have seen that the linear programming approach to this problem allows us to recover the condition (Γ), but it also reveals additional information about our continuous problem. As mentioned previously, for any dual optimisers $(\nu_j^*, \eta_{j,t}^*) \in \mathbb{R}^{L+1} \times l^\infty$, the sequence $(\nu_j^*, \eta_{j,t}^T) \in \mathbb{R}^{L+1} \times l^\infty$, where $\eta_{j,t}^T = \eta_{j,t}^* \mathbf{1}\{t \leq T\}$, is also dual feasible when T is such that $\bar{F}_{j,t}^N = 0$ for $t \geq T$. Furthermore, this new dual solution is also optimal. Since we work on the bounded domain $[x_*, x^*]$, there exists $T^* := \min\{t : \bar{F}_{j,t}^N = 0 \forall t \geq T, \forall j\}$. Anything that happens after T^* does not affect our payoff, and we therefore have some freedom past this point. We can also see this from our proof of Theorem 3.2 if we work on $[x_*, x^*]$. For $t \geq T^*$ we have $L_\infty^K(W) - L_t^K(W) = 0$, so we have equality in the primary optimisation problem (1.5) and require the secondary problem (1.7) to get the K-cave barrier shape.

In the discrete problem this freedom arises in the following way: if $p_{j,t} > 0$ for some j and $t \geq T^*$ then we can stop mass at (j,t), decreasing our local time everywhere (so remaining primal-feasible) without affecting optimality. This allows us to prove the following.

Lemma 3.14. *Let μ_l^N and μ_r^N be the distributions embedded by our optimal $(p_{j,t})$ to the left and right of K respectively. Then for any j,*

$$j \in \operatorname{supp}(\mu_l^N) \implies \bar{r}_j^N \leq T^*.$$

In particular, this means that $\operatorname{supp}(\mu_r^N) = \operatorname{supp}(\mu^N)$.

Proof. Take $j \in \operatorname{supp}(\mu_l^N)$, so $q_{j,\bar{l}_j^N - 1} > 0$, $p_{j,\bar{l}_j^N - 1} = 0$ and suppose that $T^* < \bar{r}_j^N < \infty$, so $p_{j,\bar{r}_j^N} = 0$ and $p_{j,\bar{r}_j^N - 1} > 0$. Let $\varepsilon = \min\{q_{j,\bar{l}_j^N - 1}, p_{j,\bar{r}_j^N - 1}\}$. We define new primal variables corresponding to stopping ε of mass at $(j, \bar{r}_j^N - 1)$, and releasing ε of mass at

$(j, \bar{l}_j^N - 1)$ which we stop after one step. Let

$$\begin{aligned}
\bar{p}_{j,\bar{l}_j^N-1} &= \varepsilon, & \bar{q}_{j,\bar{l}_j^N-1} &= q_{j,\bar{l}_j^N-1} - \varepsilon, \\
\bar{p}_{j+1,\bar{l}_j^N} &= p_{j+1,\bar{l}_j^N}, & \bar{q}_{j+1,\bar{l}_j^N} &= \bar{q}_{j+1,\bar{l}_j^N} + \frac{\varepsilon}{2}, \\
\bar{p}_{j-1,\bar{l}_j^N} &= p_{j-1,\bar{l}_j^N}, & \bar{q}_{j-1,\bar{l}_j^N} &= \bar{q}_{j-1,\bar{l}_j^N} + \frac{\varepsilon}{2}, \\
\bar{p}_{j,\bar{r}_j^N-1} &= p_{j,\bar{r}_j^N-1} - \varepsilon, & \bar{q}_{j,\bar{r}_j^N-1} &= q_{j,\bar{r}_j^N-1} + \varepsilon, \\
\bar{p}_{j,r} &= p_{j,r} - \tilde{p}_{j,r} \quad \forall r > s, & \bar{q}_{j,r} &= q_{j,r} + \tilde{q}_{j,r} \quad \forall r > s, \\
\bar{p}_{j,r} &= p_{j,r} \quad \text{otherwise}, & \bar{q}_{j,r} &= q_{j,r} \quad \text{otherwise},
\end{aligned}$$

where the $\tilde{p}_{j,r}$ are defined as in Theorem 3.9 with $s = \bar{r}_j^N - 1$. We can check that these new variables are primal feasible, noting in particular that $\sum_t \bar{p}_{j,t} \leq \sum_t p_{j,t}$. Furthermore, $\bar{F}^N$ is a submartingale, and so by releasing mass at $(j, \bar{l}_j^N - 1)$ we improve our payoff: $\sum \bar{F}_{j,t}^N \bar{q}_{j,t} \geq \sum \bar{F}_{j,t}^N q_{j,t}$.

We can repeat this process until either $q_{j,\bar{l}_j^N-1} = 0$ or $p_{j,\bar{r}_j^N-1} = 0$. If $p_{j,\bar{r}_j^N-1} = 0$ first then we have moved $\bar{r}_j^N \to \bar{r}_j^N - 2$ and can repeat the above from this new value of $\bar{r}_j^N$. Similarly, if $q_{j,\bar{l}_j^N-1} = 0$ first then we have moved $\bar{l}_j^N \to \bar{l}_j^N + 2$ and can repeat the above. This will continue until either $\bar{r}_j^N \leq T^*$ or $j \notin \operatorname{supp}(\mu_l^N)$. Since we improve our payoff at each step, any optimiser must have one of these properties at each j.

If $\bar{r}_j^N = \infty$ for some $j \in \operatorname{supp}(\mu_l^N)$ then we can run the above argument with any $t > T^*$ in place of $\bar{r}_j^N$ and come to the same conclusion. In particular we have a right-hand barrier whenever $j \in \operatorname{supp}(\mu_l^N)$, and obviously also for $j \in \operatorname{supp}(\mu^N) \setminus \operatorname{supp}(\mu_l^N)$. $\square$

The conclusion $\operatorname{supp}(\mu_r^N) = \operatorname{supp}(\mu^N)$ means that we have a right-hand barrier at any atom of μ^N, so in particular $\tilde{\eta}^*_{j,\bar{r}_j^N} = 0$ for all $j \in \operatorname{supp}(\mu^N)$. Then, for any $x \in \operatorname{supp}(\mu)$,

$$\Gamma(x) = \lim_{N\to\infty} \left(-\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} + \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}_x^N} + \bar{F}^N_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} \right) = \lim_{N\to\infty} -\eta^*_{\lfloor\sqrt{N}x\rfloor,\bar{l}_x^N} \leq 0.$$

In particular, (Γ) now becomes

$$\begin{aligned}
\Gamma(x) &\leq 0 \quad \mu\text{-a.s.} \\
\Gamma(x) &= 0 \quad \mu_l\text{-a.s.}
\end{aligned} \tag{Γ'}$$

and we have proved the following.

Theorem 3.15. *For a K-cave stopping time τ given by curves l, and r, the condition*

(Γ') *is both necessary and sufficient for optimality.*

3.6 Uniqueness

In Theorem 3.2 we proved that there is a K-cave barrier whose stopping time solves (OptSEP), however we also argued in Section 3.2.2 that there are multiple K-cave barriers solving (SEP). We have now characterised the optimal solutions and can ask if there are multiple K-cave barriers that are also optimal.

Similarly to Loynes [1970] we define regular K-cave barriers. Take a K-cave barrier $\mathcal{R}$ with boundary curves l and r. Recall that we define x^* to be the smallest x such that $\mu((x,\infty)) = 0$, and x_* the largest x such that $\mu((-\infty, x_*)) = 0$.

Definition 3.16. The K-cave barrier $\mathcal{R}$ is *regular* if

- l is increasing on $\{x > 0\}$, and decreasing on $\{x < 0\}$
- $l(x) = K(x) = r(x)$ for all $x \notin [x_*, x^*]$ (where l and K exist),
- $r(x) = 0$ (or $r(x) = l(x_*)$) for all $x < x_* \wedge \frac{1}{\beta} \ln k$.

Theorem 3.17. *There is a unique regular K-cave barrier whose stopping time solves* (OptSEP)*.*

Proof. Suppose τ, σ are both optimisers of (OptSEP) and hitting times of K-cave barriers with continuation regions $\mathcal{D}^\tau$ and $\mathcal{D}^\sigma$ respectively. By Theorem 3.12 these stopping times have dual optimisers G^τ, H^τ and G^σ, H^σ, where the functions G^τ, G^σ take the form $G(x,t) = -\int_t^{r(x)} M(x,s)\mathrm{d}s$ for the corresponding r and M. Then,

$$\begin{aligned}
\mathbb{E}\left[F(W_\tau,\tau)\right] &= \mathbb{E}\left[G^\tau(W_\tau,\tau) + H^\tau(W_\tau)\right] \\
&= G^\tau(W_0,0) + \int H^\tau(x)\mu(\mathrm{d}x) \\
&\geq \mathbb{E}\left[G^\tau(W_\sigma,\sigma) + H^\tau(W_\sigma)\right] \\
&\geq \mathbb{E}\left[F(W_\sigma,\sigma)\right],
\end{aligned}$$

since $G^\tau(W_t,t)$ is a supermartingale, and $G^\tau(x,t) + H^\tau(x) \geq F(x,t)$ everywhere. However, since both stopping times are optimisers, $\mathbb{E}\left[F(W_\tau,\tau)\right] = \mathbb{E}\left[F(W_\sigma,\sigma)\right]$, and we have equality in the above, so

$$\mathbb{E}\left[F(W_\sigma,\sigma)\right] = \mathbb{E}\left[G^\tau(W_\sigma,\sigma) + H^\tau(W_\sigma)\right].$$

In Section 3.4 we argue that $G^\tau(x,t) + H^\tau(x) \geq F(x,t)$ since, if we define $M^\tau(x,s) := \mathbb{E}^{x,s}\left[\partial_t^- F\left(W_\tau,\tau\right)\right]$, we have

$$\begin{aligned} t < K(x) \quad &\implies \quad F_t(x,t) = -\frac{\beta^2}{2}h(x,t) \leq M^\tau(x,t) \\ t > K(x) \quad &\implies \quad F_t(x,t) = 0 \geq M^\tau(x,t). \end{aligned}$$

It is easy to see that these inequalities are strict on $\mathcal{S}^\tau_{\mathcal{D}} := \left\{(x,t) \in \mathcal{D}^\tau : \mathbb{P}^{x,t}(F(W_\tau,\tau) > 0) > 0\right\}$, and so $G^\tau(W_\sigma,\sigma) + H^\tau(W_\sigma) > F(W_\sigma,\sigma)$ for $(W_\sigma,\sigma) \in \mathcal{S}^\tau_{\mathcal{D}}$. Therefore $(W_\sigma,\sigma) \notin \mathcal{S}^\tau_{\mathcal{D}}$ almost surely, and similarly $(W_\tau,\tau) \notin \mathcal{S}^\sigma_{\mathcal{D}}$ almost surely. The inverse barriers given by l^τ and l^σ must then coincide. To see this, suppose for contradiction that $l^\tau(x) < l^\sigma(x)$ for some x. Since our barriers are regular we must have $l^\tau(W_\sigma) < \sigma < K(W_\sigma)$ with positive probability (any increasing section or atom of l^σ will be hit with positive probability), so $\mathbb{P}\left((W_\sigma,\sigma) \in \mathcal{S}^\tau_{\mathcal{D}}\right) > 0$, which is a contradiction.

Furthermore, the argument of Loynes [1970] proves that for a given inverse barrier bounded by l^τ, there is a unique barrier given by some r^τ that gives the correct embedding. This argument runs as follows: suppose we have two Root barriers $\mathcal{R}_1$ and $\mathcal{R}_2$ given by curves r_1 and r_2 respectively. If our inverse barrier is $\mathcal{R}$, then $W_{\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_1}} \sim \mu$ and $W_{\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_2}} \sim \mu$. We can consider $\mathcal{R}_0 = \mathcal{R}_1 \cup \mathcal{R}_2$, or $r_1 \wedge r_2$, and show that $W_{\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}} \sim \mu$ also. By taking the union of the two barriers we increase the area of the stopping region and therefore ensure that no extra paths can be embedded at $\mathcal{R}$. Also, if we have points $\underline{x}$, $\bar{x}$ such that $r_1(x) \leq r_2(x)$ on $(\underline{x},\bar{x})$, then less mass is embedded in $(\underline{x},\bar{x})$ by $\mathcal{R}_0$ than $\mathcal{R}_1$, so overall less mass is embedded in $(\underline{x},\bar{x})$ by $\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}$ than $\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_1}$. Similarly at points where $r_2(x) \leq r_1(x)$ we have that less mass is embedded by $\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}$ than $\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_2}$. Then on any interval A, $\mathbb{P}\left(W_{\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}} \in A\right) \leq \mu(A)$, but since both of these distributions are probability measures we must in fact have equality.

This shows that $\mathcal{R}_0$ also embeds the correct distribution, so $W_{\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}} \sim \mu$, and therefore $\mathbb{E}\left[\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_1}\wedge\tau_{\mathcal{R}_2}\right] = \mathbb{E}\left[\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}\right] = \mathbb{E}\left[W^2_{\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_0}}\right] = \int x^2\mu(\mathrm{d}x) = \mathbb{E}\left[\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_1}\right]$, so $\tau_\mathcal{R}\wedge\tau_{\mathcal{R}_1}\wedge\tau_{\mathcal{R}_2} = \tau_\mathcal{R}\wedge\tau_{\mathcal{R}_1}$ almost surely. We can then conclude that $\mathcal{R}_1$ and $\mathcal{R}_2$ are equivalent as in Loynes [1970]. $\square$

We now summarise what we know of the uniqueness of these barriers:

- There may be many (regular) K-cave barriers whose hitting times solve (SEP).
- There is exactly one regular K-cave barrier whose hitting time solves (OptSEP), and this is the regular K-cave barrier satisfying (Γ').
- All other solutions of (OptSEP) have the same stopping region as the regular K-cave barrier solution, τ, on $\mathcal{S}^\tau = \left\{(x,t) : \mathbb{P}^{x,t}\left(F(W_\tau,\tau) > 0\right) > 0\right\}$. In particular they have the same inverse barrier.

In the spirit of Loynes [1970] we could say that two stopping regions are τ-equivalent if they agree on $\mathcal{S}^\tau$, and then any region whose hitting time solves (OptSEP) is τ-equivalent to the K-cave optimiser τ.

3.7 The Root, Rost, and cave embeddings

Given that the Root, Rost, cave, and K-cave embeddings are closely related, it seems natural that we may be able to extend the results of this chapter to the other embeddings, and in this section we give ideas on how this can be done for certain payoffs of the correct form.

The dual superhedging functions G, H are given in Cox and Wang [2013a,b] for the Root and Rost embeddings respectively, and the proofs of the optimality of these are similar to that of Theorem 3.5. Likewise we can propose such functions for the cave embedding, so now consider $F(t) = -\varphi(t)$ for φ a cave-type function and a cave barrier $\mathcal{R}$ with boundary curves l and r and stopping time τ that embeds μ. Define

$$\begin{aligned}
M(x,t) &:= \mathbb{E}^{x,t}\left[F'(\tau)\right], \\
G(x,t) &:= -\int_t^{t_0} M(x,s)\mathrm{d}s - Z(x), \\
H(x) &:= F(r(x)) - \int_{t_0}^{r(x)} M(x,s)\mathrm{d}s + (\Gamma(x))_+ + Z(x), \\
\Gamma(x) &:= \int_{l(x)}^{r(x)} \left(M(x,s) - \partial_t F(s)\right)\mathrm{d}s,
\end{aligned}$$

for some Z chosen to ensure that $G(W_t,t)$ is a martingale in the continuation region. Then we can prove the following result using Theorem 3.5.

Theorem 3.18. *If* (Γ) *holds, then* τ *is a solution of* (OptSEP) *when* $F(t) = -\varphi(t)$, *and*

$$\mathbb{E}\left[F(\tau)\right] = \mathbb{E}\left[G(W_\tau, \tau) + H(W_\tau)\right].$$

For the necessity of the condition (Γ) we can use the convergence arguments of Section 3.5.2. Note that for any payoff $F(t)$ of the Root, Rost, or cave forms, all results from Chapter 2 hold provided that F is increasing (which is necessary to ensure that we embed the full distribution μ^N in the discrete problems and therefore in the limit our barrier embeds μ). If F is differentiable and $f(t) = F'(t)$ is bounded, then we can consider instead the increasing function $F(t)+Ct$ for some constant C. The key result in the dual problem of the K-cave embedding is that we can cut-off our dual optimisers to ensure that they are in the unweighted space, l^∞, due to the fact that our payoff is constant after some fixed time. Since the function $F(t) + Ct$ may now be strictly increasing, this no longer works. We could therefore consider $F^n(t) = \max\{n, F(t)+Ct\}$ for $n \in \mathbb{N}$ with discrete counterpart $\bar{F}^{n,N}_t = F^n\left(\frac{t}{N}\right)$. This is still increasing, but there is now a finite time after which the function is constant so we can use the same cut-off argument as in the K-cave problem.

For each N, this mechanism gives dual optimisers $(\eta^N, \nu^N) \in l^\infty \times \mathbb{R}^{L+1}$, so Lemma 3.10 applies to show that, for $\tilde{\eta}^N_{j,t} = \eta^N_{j,t} + \bar{F}^{n,N}_t$,

$$\tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} - \tilde{\eta}^*_{\lfloor\sqrt{N}x\rfloor,\bar{r}^{n,N}_x} \to \int_{r^n(x)}^t \mathbb{E}^{x,s}\left[\partial_t F^n(\tau)\right]\mathrm{d}s \quad \text{as } N\to\infty,$$

where r^n is the curve bounding the Root barrier, the Rost inverse barrier, or the (Root) barrier part of the cave barrier and $\bar{r}^{n,N}$ is the discrete equivalent. We no longer have $\bar{F}^{n,N}_{\bar{r}^{n,N}_x} = 0$ for each $x \in \operatorname{supp}(\mu_r)$, so we find that

$$\lim_{N\to\infty} \tilde{\eta}^N_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} = -\int_t^{r^n(x)} M^n(x,s)\mathrm{d}s + F^n(r^n(x)) + (\Gamma(x))_+,$$

where

$$\begin{aligned}
M^n(x,t) &:= \mathbb{E}^{x,t}\left[\partial_t F^n(\tau)\right] \\
\Gamma^n(x) &:= \int_{l^n(x)}^{r^n(x)} \left(M^n(x,s) - \partial_t F^n(s)\right)\mathrm{d}s \\
&= \int_{l^n(x)}^{r^n(x)} M^n(x,s)\mathrm{d}s - \left(F^n(r^n(x)) - F^n(l^n(x))\right)
\end{aligned}$$

and l^n is identically 0 for the Root and Rost barriers, and is the curve bounding the (Rost) inverse barrier part of the cave barrier. We must also have that Γ^n satisfies (Γ)

by Theorem 3.12. Then

$$\lim_{N\to\infty} \tilde{\eta}^N_{\lfloor\sqrt{N}x\rfloor,\lfloor Nt\rfloor} = G^n(x,t) + H^n(x),$$

where, in the case of the cave embedding,

$$\begin{aligned}
G^n(x,t) &:= -\int_t^{t_0} M^n(x,s)\mathrm{d}s - Z^n(x) \\
H^n(x) &:= F^n(r^n(x)) - \int_{t_0}^{r^n(x)} M^n(x,s)\mathrm{d}s + (\Gamma^n(x))_+ + Z^n(x),
\end{aligned}$$

for Z^n chosen so that $G^n(W_t,t)$ is a martingale. In the Root and Rost cases we recover the G and H from Cox and Wang [2013a] and Cox and Wang [2013b] respectively.

For any n we then have

$$\mathbb{E}\left[F^n(\tau^n)\right] = \mathbb{E}\left[G^n(W_{\tau^n},\tau^n) + H(W_{\tau^n})\right]$$

where each τ^n is the hitting time of a barrier of the correct form with barriers bounded by l^n and r^n in the cave embedding case. From the compactness result of Root [1969], as used in Lemma 2.19, these barriers converge to some barrier with stopping time τ, and $(W_{\tau^n},\tau) \xrightarrow{\mathbb{P}} (W_\tau,\tau)$ as $n\to\infty$. Let $M(x,t) := \mathbb{E}^{x,t}\left[f(\tau)\right]$ and suppose that f is bounded. Then by the dominated convergence theorem, $M^n(x,t)\to M(x,t)$ for each (x,t) along some subsequence. Furthermore, in the cave embedding case,

$$\begin{aligned}
G^n(x,t) \to G(x,t) &:= -\int_t^{t_0} M(x,s)\mathrm{d}s - Z(x) \\
H^n(x) \to H(x) &:= F(r(x)) - \int_{t_0}^{r(x)} M(x,s)\mathrm{d}s + (\Gamma(x))_+ + Z(x) \\
\Gamma^n(x) \to \Gamma(x) &:= \int_{l(x)}^{r(x)} \left(M(x,s) - \partial_t F(s)\right)\mathrm{d}s,
\end{aligned}$$

for the appropriate Z. In particular, Γ satisfies (Γ) and so by Theorem 3.13, G and H are optimal:

$$\mathbb{E}\left[F(\tau)\right] = \mathbb{E}\left[G(W_\tau,\tau) + H(W_\tau)\right].$$

3.8 Further Work

In this chapter we have presented a full characterisation of the K-cave embedding in the case of one dimensional (geometric) Brownian motion and, as with any embedding, there are many natural extensions and modifications of the problem to consider. Some extensions, such as considering a more general process, should not present anymore difficulty in the case of the cave and K-cave embeddings than those of Root and Rost, however there are some questions which seem much more demanding in these cases.

Consider maximising the payoff

$$F(x,t) = \left(x^\beta \exp\left(-\frac{\beta(\beta-1)}{2}t\right) - k_1\right)_+ + \left(x^\beta \exp\left(-\frac{\beta(\beta-1)}{2}t\right) - k_2\right)_+$$

for $0 < k_1 < k_2$. In the case of the Root embedding we can add two concave functions and still have a concave payoff, so the optimal stopping region will remain a barrier. When considering the payoff F, the stop-go argument will indeed ensure that any optimiser has an inverse-barrier in the region $t < K_2(x)$ and a barrier for $t > K_1(x)$, but it is less clear what happens between the curves K_1 and K_2. If no stopping is allowed in this region then it is not clear that there is an embedding of this form for given k_1, k_2, and μ.

One generalisation considered for previous embeddings is the multi-marginal embedding version of the problem, as introduced earlier. In this case we may hope to find a sequence of K-cave barriers as in the case of the Root embedding from Cox et al. [2018], but with the added difficulty of the inverse-barrier. Two different approaches have been used in recent papers to consider the multi-marginal case. In Cox et al. [2018], the authors construct consecutive stopping regions for the Root embedding by solving an iterated sequence of optimal stopping problems. The optimal stopping problem is a probabilistic interpretation of a variational inequality found in Cox and Wang [2013a] and Gassiat et al. [2015]. Similar problems can be formulated in the case of the Rost embedding, see McConnell [1991], Cox and Wang [2013b], Gassiat et al. [2015], De Angelis [2015]. It may be possible to find such a stopping problem in the case of the cave or K-cave embeddings. The cave embedding seems the easier of the two in this situation, and the correct formulation may switch between problems similar to the Root and Rost situations for $\tau < t_0$ and $\tau > t_0$.

More recently, the monotonicity principle has been extended to the multi-marginal case in Beiglböck et al. [2017a]. In short, the authors again consider running two paths of a Brownian motion and then swapping their tails at a possible stopping time of one of the paths, and examining the the values of the payoff functions before and after this swap. In the multi-marginal case however there are many stopping times, and we therefore have the option to swap the paths back again at later stopping times, and these different swapping rules are called colour swaps. The authors show that in the case of the classical embeddings of e.g. Azéma-Yor, Root, Rost, Vallois, it suffices to consider one specific swapping rule, namely swapping the paths back as soon as possible. This rule does not work in simple examples of the cave or K-cave embeddings, since we find that the set of stop-go pairs is empty. An interesting question is then to find the correct set of stop-go pairs in this problem when general colour swaps are considered.

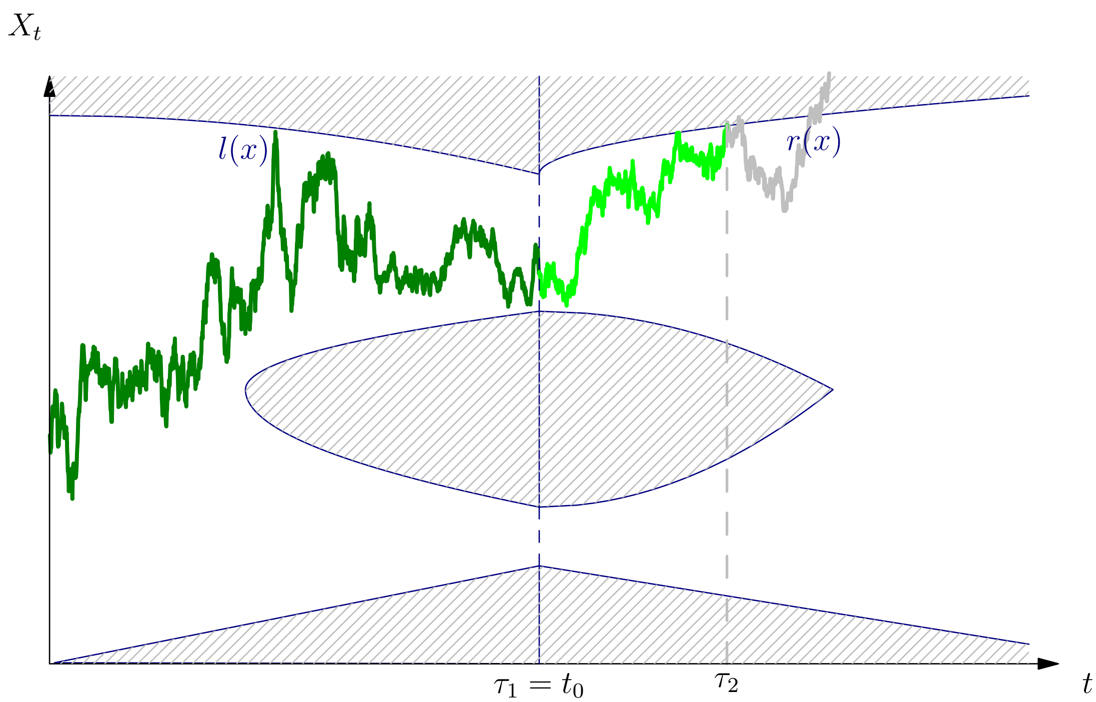

Figure 3-4: An example inverse cave barrier with randomised stopping across the dashed line. Some paths are stopped at t_0, others continue to τ_2.

Finally, we have considered superhedging our European option, but we may also wish to subhedge the option to find a model-independent lower bound on the price. The stop-go argument works exactly in reverse, showing that if (x, t) is in our stopping region for $t < K(x)$, then so is (x, s) for any $t \leq s \leq K(x)$, and similarly for $t > K(x)$. We then have a Root barrier in $\{t \leq K(x)\}$ and a Rost inverse barrier in $\{t \geq K(x)\}$. However, it is impossible to find an embedding of this form unless we allow the process to cross the curve K without stopping. If μ has full support then we cannot have regions of K which do not embed any mass, so we must have some randomised stopping across the curve K. We call such a region an inverse cave, or K-cave, barrier, see Figure 3-4. If l denotes the boundary of the barrier, and r the boundary of the inverse-barrier, then these randomised regions occur when $l = K = r$, and we suspect that this will be enforced through a condition similar to (Γ). An equivalent proof of Theorem 3.5 is dependent on a suitable choice of G^* to ensure that $G^*(W_t, t)$ is a supermartingale in the continuation region. There are few examples of true randomised stopping time solutions to (OptSEP) in the literature, however the problem fits easily into the framework of Chapter 2 where we allow points (j, t) such that $p_{j,t}q_{j,t} > 0$.

Chapter 4

Continuous-Time Optimisation

In this chapter we again reformulate (OptSEP) as a deterministic optimisation problem, optimising over the possible continuation measures of feasible stopping rules. This can be seen as a continuous-time version of the discrete problem of Chapter 2, and many of the results are similar. We are able to prove a strong duality result which gives the existence of dual optimisers in a certain weighted space.

As with the discrete approach of Chapter 2, this problem is robust in that we can consider arbitrary (smooth) starting distributions. The constraints in the dual problem also suggest the form of the dual variables, and we could use this to recover, for example, the superhedging portfolio of Chapter 3.

4.1 Introduction and Notation

The methods of Chapter 2 allow us to prove a duality result for a discrete optimisation problem, and then limiting arguments are necessary to transfer these results to (OptSEP). In this chapter we avoid this limiting argument by remaining in the continuous time setup, but as in Chapter 2 we reformulate (OptSEP) so that the optimisation is over deterministic functions, thus removing the need for heavy probabilistic machinery. We see that this problem is much more difficult than the discrete version, but the two approaches share some features. One important similarity is the apparent necessity of requiring some exponential decay of the number of remaining Brownian paths, and we discuss this in Section 4.6.

Let ν, μ be probability measures on $\mathbb{R}$ such that ν has bounded density $\rho_0(x)$ and $\nu \preceq_c \mu$. Define $x^* = \inf\{x : \mu((x,\infty)) = 0\}$ and $x_* = \sup\{x : \mu((-\infty,x)) = 0\}$, where $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$, and note that the convex ordering ensures that $\mathrm{supp}(\rho_0) \subseteq [x_*, x^*]$. We restrict to the case where $-\infty < x_* < x^* < \infty$, so the initial and target measures have compact support, however most arguments can be carried over to the infinite case with sufficient control on the payoff function. We will work in $[x_*, x^*] \times [0,\infty)$ throughout this chapter.

Consider running a Brownian motion B with $B_0 = 0$ up to the stopping time $H_{x_*}(B) \wedge H_{x^*}(B)$, where $H_x(B) = \inf\{t \geq 0 : B_t = x\}$. The paths of this stopped process have some continuation density $\pi(x,t)$, for example if $x_* = -\infty$ and $x^* = \infty$, then $\pi(x,t) = \frac{1}{2\pi t}\mathrm{e}^{-\frac{x^2}{2t}}$, the usual Brownian transition density.

Suppose we have another Brownian motion W with $W_0 \sim \nu$, which we run up to the stopping time $H_{x_*}(W) \wedge H_{x^*}(W)$. The paths of this process have density

$$\rho(x,t) := \int_{x_*}^{x^*} \pi(x-y,t)\rho_0(y)\mathrm{d}y.$$

Define also $\bar{\rho}(x,t) = \rho(x,t)\mathrm{e}^{-\lambda t}$, for some $\lambda > 0$, which corresponds to stopping W exponentially up to the stopping time $H_{x_*}(W) \wedge H_{x^*}(W)$. We choose λ so that $\frac{\int_0^\infty \bar{\rho}(\cdot,t)\mathrm{d}t}{L_\mu^\cdot} \in L^\infty[x_*, x^*]$, where $L_\mu^x := U_\nu(x) - U_\mu(x)$ is the expected local time accrued at x by a stopped process embedding μ with initial distribution ν. We also have $\bar{\rho} \in L^\infty$ since ρ_0 is bounded.

Note that

$$\begin{aligned}
\frac{1}{2}\rho_{xx}(x,t) - \rho_t(x,t) &= \int_{x_*}^{x^*} \left(\frac{1}{2}\pi_{xx}(x-y,t) - \pi_t(x-y,t)\right)\rho_0(y)\mathrm{d}y = 0, \\
\frac{1}{2}\bar{\rho}_{xx}(x,t) - \bar{\rho}_t(x,t) &= \left(\frac{1}{2}\rho_{xx}(x,t) - \rho_t(x,t)\right)\mathrm{e}^{-\lambda t} + \lambda\bar{\rho}(x,t) = \lambda\bar{\rho}(x,t),
\end{aligned}$$

for $(x,t) \in (x_*, x^*) \times (0,\infty)$.

In the following we set up an infinite linear optimisation problem, optimising over continuation measures, denoted p, and stopping measures, q, of paths of a stopped Brownian motion in $[x_*, x^*]$. With this intuition in mind, we have certain properties of p and q, for example any feasible p is dominated by the measure with density $\rho(x,t)$, and therefore we can think of feasible p as densities in some function space. To include barrier-type solutions of (OptSEP) we need to include stopping measures which are not absolutely continuous with respect to Lebesgue, but any feasible stopping distribution

q should be a measure with $\|q\|_{TV} = 1$. As in Chapter 2, we optimise only over the continuation densities, setting $q = \frac{1}{2}p_{xx} - p_t$, to be interpreted in a distributional sense. Note that this exists since we always take locally integrable p, so any p defines a distribution. In the definition of the distributional derivative, since we are working on $[x_*, x^*] \times \mathbb{R}^+$, we consider test functions in $C_0^\infty((x_*, x^*) \times (0, \infty))$. Any positive distribution (mapping any positive test function to a positive real) has a corresponding positive Radon measure, and so we will restrict to the case where the distribution given by $\frac{1}{2}p_{xx} - p_t$ defines a measure. In some cases we will assume further that $\frac{1}{2}p_{xx} - p_t$ exists weakly, in which case the stopping distribution has a density, so we take q to be the density function $q(x, t)$.

For distributions T_1, T_2 on a set S, we say that $T_1 \leq T_2$ *weakly*, if for all test functions $\varphi \in C_0^\infty(\mathsf{S})$ with $\varphi \geq 0$ we have

$$\langle T_1, \varphi \rangle \leq \langle T_2, \varphi \rangle,$$

and we write $T_1 \leq_{w(\mathsf{S})} T_2$. Note that any locally-integrable function, f, and (finitely-additive) measure, σ, have corresponding distributions T_f, T_σ respectively where

$$\langle T_f, \varphi \rangle = \int_{\mathsf{S}} \varphi(x) f(x) \mathrm{d}x,$$
$$\langle T_\sigma, \varphi \rangle = \int_{\mathsf{S}} \varphi(x) \sigma(\mathrm{d}x),$$

for any $\varphi \in C_0^\infty(\mathsf{S})$. We abuse notation by writing, for example, $f \leq_{w(\mathsf{S})} \sigma$ to mean $T_f \leq_{w(\mathsf{S})} T_\sigma$.

Recall that for a distribution, T, defined on an open set S, given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with corresponding partial derivative operator ∂^α, the derivative $\partial^\alpha T$ is the distribution on S such that for any $\varphi \in C_0^\infty(\mathsf{S})$,

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle.$$

Note that if we have $p : [x_*, x^*] \times [0, \infty) \to \mathbb{R}$ then the derivative of T_p defines a distribution on $(x_*, x^*) \times (0, \infty)$, however by considering the extension $\bar{p} : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ with $\bar{p}(x, t) = \mathbf{1}\{x \in (x_*, x^*)\} p(x, t)$ we can define the derivatives on $[x_*, x^*] \times (0, \infty)$.

To prove the strong duality result Theorem 2.5 in Chapter 2 we weighted the measures with an exponential weighting, and in the continuous setup this is the role of the functions $\bar{\rho}$ introduced above. Since some of our stopping distributions do not have densities, we need a way to weight measures. For a (signed) measure ξ on a set $\mathsf{S} \subseteq \mathbb{R}^n$ and a function $f : \mathsf{S} \to \mathbb{R} \setminus \{0\}$, we write $\frac{\xi}{f}$ to denote the (signed) measure ζ on S such that for any Borel $A \subseteq \mathsf{S}$, $\zeta(A) = \int_A \frac{\xi(\mathrm{d}x,\mathrm{d}t)}{f(x,t)}$. We also denote $f \cdot \xi := \frac{\xi}{(f)^{-1}}$.

It is useful at this stage for the reader to have an idea of the type of problem we will be considering. Our primal problem will take the form

$$\begin{aligned}\sup_p \Phi(p) :&= \left\langle F, \frac{1}{2}p_{xx} - p_t \right\rangle + \int F(x,0)\left(\rho_0(x) - p(x,0)\right)\mathrm{d}x \\ &= \left\langle \frac{1}{2}F_{xx} + F_t, p \right\rangle + \int F(x,0)\rho_0(x)\mathrm{d}x\end{aligned}$$

over functions $p(x,t)$ subject to

- $p \in \mathsf{X}$
- $p(x,t) \geq 0, \qquad \forall x \in [x_*, x^*], t \geq 0$
- $\frac{1}{2}p_{xx} - p_t \geq_{w(\mathsf{S})} 0,$
- $\int_0^\infty p(x,t)\mathrm{d}t \leq L_\mu^x \qquad \forall x \in [x_*, x^*]$
- $p(x,0) \leq \rho_0(x), \qquad \forall x \in \operatorname{supp}(\nu),$

and the possible choices of X will be defined in the next section. We take derivatives in the distributional sense.

In full generality we should optimise over continuation measures p defined on $\mathbb{R} \times [0,\infty)$, however to ensure that we consider only UI embeddings we need to impose the conditions $p(x,t) = 0$ for $x \leq x_*$ or $x \geq x^*$, and these are enforced by the local time condition. Note that $\rho(x,t)$ and $\bar{\rho}(x,t)$ have this property, and since we will be considering $\frac{p}{\bar{\rho}}$, to avoid division by zero we restrict the continuation measures p to be defined on (x_*, x^*), rather than taking the extension $\bar{p}$.

Allowing $p(x,0) < \rho_0(x)$ corresponds to admitting a solution stopping mass immediately. It is well known for example that in the case of the Rost embedding, if $\mu\left(\overline{\operatorname{supp}(\nu)}\right) > 0$ then some randomised stopping is required at time 0.

Note that our problem is of the form

$$\sup_p \Phi(p) \quad \text{over } p \in \mathsf{X}$$

subject to

- $p \geq 0$
- $Ap \geq B$,

where Φ is linear, and A, B are given by

$$Ap = \begin{pmatrix} \frac{1}{2}p_{xx} - p_t \\ (-p(x,0))_{x \in \text{supp}(\nu)} \\ \left(-\int_0^\infty p(x,t)\mathrm{d}t\right)_{x \in [x_*, x^*]} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ (-\rho_0(x))_{x \in \text{supp}(\nu)} \\ \left(-L_\mu^x\right)_{x \in [x_*, x^*]} \end{pmatrix}.$$

Since the distributional derivative is a linear operator, A is a linear map.

4.2 Banach Spaces and Functional Analysis

In this section we state more carefully the spaces in which our distributions p, q will live and examine the corresponding dual spaces. We first state results from functional analysis which are needed later.

4.2.1 Common Banach Spaces and their Duals

In this section we let S denote a set, equipped with some topology, with a Borel σ-algebra Σ.

Definition 4.1. For a measure ξ on (S, Σ) and any $1 \leq p \leq \infty$, define a norm $\|\cdot\|_p$ on the set of measurable functions $f : \mathsf{S} \to \mathbb{R}$ by

$$\|f\|_p := \begin{cases} \left(\int_{\mathsf{S}} |f(x)|^p \xi(dx)\right)^{\frac{1}{p}}, & \text{if } 1 \leq p < \infty, \\ \inf\left\{t \middle|\, \xi\left(\{x \in \mathsf{S} :\ |f(x)| > t\}\right) = 0\right\}, & \text{if } p = \infty. \end{cases}$$

Then $L^p(\mathsf{S}; \xi)$ is the space of real-valued measurable functions on S with finite $\|\cdot\|_p$ norm, or more precisely, the equivalence classes of such functions where f and g are equivalent if $f(x) = g(x)$ ξ-almost everywhere. For $1 \leq p \leq \infty$, $L^p(\mathsf{S}; \xi)$ is a Banach space.

We write $L^p(\mathsf{S}) := L^p(\mathsf{S}; \mathrm{Leb})$ when the measure ξ is the Lebesgue measure. Also, for $\mathsf{U} \subseteq \mathsf{S}$, we write $f \in L^p(\mathsf{U})$ to mean that the restriction of f to U is in $L^p(\mathsf{U})$, i.e. $f|_{\mathsf{U}} \in L^p(\mathsf{U})$.

Equivalently, $L^p(\mathsf{S})$ can be thought of (up to equivalence classes) as the completion under the above norm of the space of real-valued continuous functions on S with compact support.

Recall that for a vector space V over a field K, the continuous dual space V^* is the set of continuous linear functionals $F : V \to K$. It is well known that for σ-finite S and $1 \le p < \infty$, $L^p(\mathsf{S})^*$ is isomorphic to $L^q(\mathsf{S})$, for q such that $\frac{1}{p} + \frac{1}{q} = 1$, where for $p = 1$ we take $q = \infty$. The isomorphism relates $g \in L^q(\mathsf{S})$ to some $F_g \in L^p(\mathsf{S})^*$ by defining a functional such that

$$F_g(f) = \int_{\mathsf{S}} f(x)g(x)dx, \quad \text{for all } f \in L^p(\mathsf{S}).$$

We will write $L^p(\mathsf{S})^* = L^q(\mathsf{S})$.

For the dual of $L^\infty(\mathsf{S})$ we require the following definitions.

Definition 4.2. For a (signed) measure ξ on a measurable space (S, Σ), a set $\mathsf{U} \in \Sigma$ is inner regular if

$$\xi(\mathsf{U}) = \sup\{\xi(\mathsf{E}) : \ \mathsf{U} \supseteq \mathsf{E}, \mathsf{E} \in \Sigma \text{ compact}\},$$

and outer regular if

$$\xi(\mathsf{U}) = \inf\{\xi(\mathsf{E}) : \ \mathsf{U} \subseteq \mathsf{E}, \mathsf{E} \in \Sigma \text{ open}\}.$$

The measure ξ is outer (inner) regular if every measurable set is outer (inner) regular, and ξ is called regular if it is both outer and inner regular.

Definition 4.3. For a space S with Σ the σ-algebra of Borel sets, we define

$$\begin{aligned} ba(\mathsf{S}) &:= \{\text{bounded, finitely additive, signed measures on } \Sigma\}, \\ ca(\mathsf{S}) &:= \{\text{bounded, countably additive, signed measures on } \Sigma\}, \end{aligned}$$

and when S is a topological space,

$$rba(\mathsf{S}) := \{\text{regular, bounded, finitely additive, signed measures on } \Sigma\},$$
$$rca(\mathsf{S}) := \{\text{regular, bounded, countably additive, signed Borel measures on } \Sigma\},$$

each equipped with the total variation norm.

Immediately we note the inclusions $rca(\mathsf{S}) \subseteq ca(\mathsf{S}) \subseteq ba(\mathsf{S})$. Finitely-additive measures are notoriously difficult to work with, however there are various results, see for example Dunford et al. [1957], Kakutani [1941], Yosida and Hewitt [1952], which shed some light on the space $ba(\mathsf{S})$. For example, we can move from any topological space S to a compact Hausdorff space Ω using the Stone-Čech compactification. We can then relate, for example, $rba(\mathsf{S})$ to the space of σ-additive measures on some compact Hausdorff space. We will not consider these results in detail here, and instead we consider how these spaces may arise as dual spaces of more simple Banach spaces.

Theorem 4.4. *[See for example Dunford et al. [1957]] The dual space $L^\infty(\mathsf{S})^*$ is isomorphic to*

$$\{\nu \in ba(\mathsf{S}) : \nu \text{ absolutely continuous w.r.t Leb}\}.$$

For any *countably* additive signed measure ν absolutely continuous with respect to Lebesgue, the Radon-Nikodym theorem shows the existence of a function $f \in L^1(\mathsf{S})$ such that $\nu(\mathsf{U}) = \int_{\mathsf{U}} f(x)\mathrm{d}x$ for any $\mathsf{U} \subseteq \mathsf{S}$. Furthermore, this map defines an isometry from $\{\nu \in ca(\mathsf{S}) : \nu \text{ absolutely continuous w.r.t Leb}\} \subseteq L^\infty(\mathsf{S})^*$ into $L^1(\mathsf{S})$, and so we have

$$L^1(\mathsf{S}) \subseteq L^1(\mathsf{S})^{**} = L^\infty(\mathsf{S})^*.$$

These spaces also arise as the duals of other common Banach spaces.

Theorem 4.5. *[See for example Dunford et al. [1957]] Let $C_b(\mathsf{S})$ be the set of real-valued, bounded, continuous functions on a topological space S with the $\|\cdot\|_\infty$ norm. Then*

$$C_b(\mathsf{S})^* = rba(\mathsf{S}).$$

Let $C_c(\mathsf{S})$ the set of real-valued, continuous functions on S that have compact support, again with the norm $\|\cdot\|_\infty$. If S is compact, then this defines a Banach space, and

$$C_c(\mathsf{S})^* = rca(\mathsf{S}).$$

If S is not compact, but is locally compact, then we can obtain a Banach space by setting

$C_0(\mathsf{S}) = \overline{C_c(\mathsf{S})}$, *and then*

$$C_0(\mathsf{S})^* = rca(\mathsf{S}).$$

Clearly the set rca is the easiest of these measure spaces to work with, however we will also be considering the dual spaces of this set of measures, so the bidual of the space of continuous functions with compact support. The second duals of the sets of continuous functions are examined in detail in Kaplan [1957, 1959], however these spaces have very few simple representations, and in the rest of this section we present results which allow us to consider elements of $rca(\mathsf{S})^*$ as members of $(\oplus_{i\in\mathcal{I}} L^\infty(\mathsf{S};\mu_i))_{l^\infty}$ for some collection $(\mu_i)_{i\in\mathcal{I}}$. For this it will be useful to recall the following definitions.

Definition 4.6. The weak topology on a topological space S is the smallest topology (fewest open sets) such that every member of the dual space S^* is continuous with respect to that topology. If (x_α) is a net in S and $x \in \mathsf{S}$, then $x_\alpha \xrightarrow{w} x$ if and only if $x^*(x_\alpha) \to x^*(x)$ for every $x^* \in S^*$.

For a normed space S, the canonical embedding Q maps $x \in \mathsf{S}$ to $Q(x) \in \mathsf{S}^{**}$, where $Q(x)(x^*) = x^*(x)$ for $x^* \in \mathsf{S}^*$. The weak-* topology is the smallest topology for S^* such that for each $x \in \mathsf{S}$, the linear functional $x^* \mapsto x^*(x)$ on S^* is continuous with respect to that topology. If (x^*_α) is a net in S^* and $x^* \in \mathsf{S}^*$, then $x^*_\alpha \xrightarrow{w^*} x^*$ if and only if $x^*_\alpha(x) \to x^*(x)$ for every $x \in \mathsf{S}$.

The map Q allows us to compare S and S^{**} in the following results from Megginson [2012], where B_S denotes the closed unit ball in the space S.

Theorem 4.7 (Goldstine)**.** *Let S be a normed space and let Q be the canonical map from S into S^{**}. Then $Q(B_\mathsf{S})$ is weakly-* dense in $B_{\mathsf{S}^{**}}$.*

Corollary 4.8. *Let S be a normed space and let Q be the canonical map from S into S^{**}. Then $Q(\mathsf{S})$ is weakly-* dense in S^{**}.*

It is clear that $Q(\mathsf{S}) \subseteq \mathsf{S}^{**}$, but the Goldstine theorem shows further that for a normed space S, $Q(\mathsf{S})$ is weakly-* dense in S^{**}. In particular then, for a set U, $L^1(\mathsf{U})$ is weakly-* dense in $L^1(\mathsf{U})^{**} = L^\infty(\mathsf{U})^*$. Therefore for any $f \in L^\infty(\mathsf{U})^*$ and $\varepsilon > 0$, there exists $f^\varepsilon \in L^1(\mathsf{U})$ such that for any $g \in L^\infty(\mathsf{U})$,

$$\left| \int_\mathsf{U} g(x) f^\varepsilon(x)\mathrm{d}x - \int_\mathsf{U} g(x) f(\mathrm{d}x) \right| < \varepsilon.$$

In particular, we can take $g(x) = \mathbf{1}_A(x)$. Similarly, $C_0(\mathsf{U})$ is weakly-* dense in $rca(\mathsf{U})^*$. Note that in general the weak-* convergence implies only the existence of a convergent net, however since we are working only with metric spaces (the dual of any Banach space is also a Banach space, and therefore a metric space) it suffices to consider sequences. We will also make use of the Hahn-Banach theorem.

Theorem 4.9 (Hahn-Banach, see for example Megginson [2012])**.** *Suppose that f_0 is a bounded linear functional on a subspace T of a normed space S. Then there is a bounded linear functional f on S such that $\|f\| = \|f_0\|$ and the restriction of f to T is f_0.*

We can use the Hahn-Banach theorem to compare the dual space of a Banach space S to that of a subspace T. Consider taking the adjoint of the identity map from S to T to get a map $\mathsf{T}^* \to \mathsf{S}^*$. If T is a closed subspace of S equipped with the same norm then the Hahn-Banach theorem implies there is an isometric isomorphism between T^* and $\mathsf{S}^*/\mathsf{T}^\perp$. If T is a dense subspace of S equipped with a stronger norm then this map is injective and we have the results of the following theorem. We cannot find a concise reference for such a result, however see for example Megginson [2012] for a discussion on the Hahn-Banach theorem and the relations between elements of T^* and equivalence classes of S^*.

Theorem 4.10. *For a Banach space S with subspace T, we have the following relations on the dual spaces:*

- *If T is a closed subspace of S equipped with the same norm, then $\mathsf{T}^* \subseteq \mathsf{S}^*$, in the sense that there is a surjection from S^* onto T^**
- *If T is a dense subspace of S equipped with a stronger norm, then $\mathsf{S}^* \subseteq \mathsf{T}^*$.*

Proof. Suppose first that T is a closed subspace of S. Then for any $f_0 \in \mathsf{T}^*$, by Hahn-Banach there is a corresponding $f \in \mathsf{S}^*$ that agrees with f_0 on T. This functional f is not necessarily unique (if T is a proper subspace of S then it won't be unique), and so $\mathsf{T}^* \subseteq \mathsf{S}^*$ in the sense that there is a surjection from S^* onto T^*.

For the second case there is some $C > 0$ such that for any $x \in \mathsf{T}$, $\|x\|_{\mathsf{S}} \leq C\|x\|_{\mathsf{T}}$. Then for any $f \in \mathsf{S}^*$ we have

$$|f(x)| \leq \|f\|_{\mathsf{S}^*}\|x\|_{\mathsf{S}} \leq C\|f\|_{\mathsf{S}^*}\|x\|_{\mathsf{T}} < \infty,$$

so $f \in \mathsf{T}^*$. Since T is dense in S, two different elements of S^* differ on T, and therefore $\mathsf{S}^* \subseteq \mathsf{T}^*$. $\square$

We can use this theorem to prove various dual space relations that will be useful later. For example, for a topological space S, we have $rca(\mathsf{S}) \subset rba(\mathsf{S})$, and both spaces are equipped with the same norm. Therefore

$$C_0(\mathsf{S})^{**} = rca(\mathsf{S})^* \subset rba(\mathsf{S})^* = C_b(\mathsf{S})^{**}.$$

From the Goldstine theorem we know that $C_0(\mathsf{S})$ and $C_b(\mathsf{S})$ are dense (in their respective weak-* topologies) in $rca(\mathsf{S})^*$ and $rba(\mathsf{S})^*$ respectively, and this gives us some understanding of these spaces, but to conclude this section we give an alternative view. The following result follows from the proof of the stronger result Albiac and Kalton [2016, Proposition 4.3.8].

Theorem 4.11. *There exists an isometric isomorphism between $rca(\mathsf{S})$ and*

$$\left(\bigoplus_{i\in\mathcal{I}} L^1(\mathsf{S};\mu_i)\right)_{l^1} := \left\{(f_i)_{i\in\mathcal{I}} :\ f_i \in L^1(\mathsf{S};\mu_i)\ \forall i \in \mathcal{I},\ \sum_{i\in\mathcal{I}} \|f_i\|_{L^1(\mathsf{S};\mu_i)} < \infty \right\},$$

the l^1-sum of spaces $L^1(\mathsf{S};\mu_i)$ for some probability measures $(\mu_i)_{i\in\mathcal{I}}$ and index set $\mathcal{I}$.

The dual space $rca(\mathsf{S})^*$ is isomorphic to

$$\left(\bigoplus_{i\in\mathcal{I}} L^\infty(\mathsf{S};\mu_i)\right)_{l^\infty} := \left\{(f_i)_{i\in\mathcal{I}} :\ f_i \in L^\infty(\mathsf{S};\mu_i)\ \forall i \in \mathcal{I},\ \sup_{i\in\mathcal{I}} \|f_i\|_{L^\infty(\mathsf{S};\mu_i)} < \infty \right\}.$$

Proof, see Albiac and Kalton [2016, Proposition 4.3.8]. By Zorn's lemma, take a maximal set of mutually singular probability measures, say $(\mu_i)_{i\in\mathcal{I}}$, on S. Then for any $\nu \in rca(\mathsf{S})$ define $(f_i)_{i\in\mathcal{I}}$ to be the functions $f_i \in L^1(\mathsf{S};\mu_i)$ such that f_i is the Radon-Nikodym derivative $\frac{\mathrm{d}\nu}{\mathrm{d}\mu_i}$, so $\mathrm{d}\nu = f_i \mathrm{d}\mu_i + \mathrm{d}\gamma$, where γ is singular with respect to μ_i. It follows from Albiac and Kalton [2016, Proposition 4.3.8] that $\nu = \sum_{i\in\mathcal{I}} f_i\mu_i$, and furthermore $\|\nu\|_{TV} = \sum_{i\in\mathcal{I}} \|f_i\|_{L^1(\mathsf{S};\mu_i)}$ so the map $\nu \mapsto (f_i)_{i\in\mathcal{I}}$ is isometric, and only countably many terms in the sum can be non-zero. The map is surjective since for any $(f_i)_{i\in\mathcal{I}}$ we can define ν by $\nu(A) = \sum_{i\in\mathcal{I}} \int_A f_i \mathrm{d}\mu_i$.

Each μ_i is a probability measure, and therefore σ-finite, so $L^1(\mathsf{S};\mu_i)^* = L^\infty(\mathsf{S};\mu_i)$. For Banach spaces $(X_i)_{i\in\mathcal{I}}$, we know that $(\oplus_{l^1} X_i)^* = \oplus_{l^\infty} X_i^*$. $\square$

The above result is important in that it allows us to consider elements of $rca(\mathsf{S})^*$ as nets of functions in $(\oplus_{i\in\mathcal{I}} L^\infty(\mathsf{S};\mu_i))_{l^\infty}$, albeit for unknown measures μ_i. We believe that this decomposition of the dual space may be related to the ideas of weak arbitrage from Davis and Hobson [2007].

In this paper the authors define a market to have a weak arbitrage if there is a portfolio with non-positive initial cost, but for which all subsequent cashflows are non-negative, with non-zero probability of a strictly positive cashflow. To see how this relates, consider the example of a market in which there are two call options with different strikes trading at the same price, so $C(K_1) = C(K_2)$ for some $0 < K_1 < K_2$. If these two options are on some underlying S and both have maturity T, then the disjoint events $\{S_T > K_1\}$, $\{S_T \leq K_1\}$ have different arbitrage strategies. We can buy the call with strike K_1 and sell the call with strike K_2 for zero cost, and then we are guaranteed a non-negative payoff, with a strict profit if $S_T > K_1$. Alternatively, we can sell the option with strike K_1 for a profit, and we retain this profit if $S_T \leq K_1$. We know there is an arbitrage, but we cannot tell at time zero which strategy attains the profit. It could be the case that the different choice of measure in the decomposition of $rca(\mathsf{S})^*$ correspond to the scenarios in which we pick different arbitrage strategies.

4.2.2 Primal and Dual Spaces

From this point we define $\mathsf{S} := (x_*, x^*) \times (0, \infty)$, $\overline{\mathsf{S}} := [x_*, x^*] \times [0, \infty)$, $\mathsf{T} := \text{supp}(\rho_0)$. Recall that for $f : \overline{\mathsf{S}} \to \mathbb{R}$ we write $f \in L^p(\mathsf{S})$ to mean that $f|_{\mathsf{S}} \in L^p(\mathsf{S})$. Similarly we denote $\|f\|_{L^\infty(\mathsf{S})} := \|f|_{\mathsf{S}}\|_{L^\infty(\mathsf{S})}$ and for a measure ξ on $\overline{\mathsf{S}}$ we write $\|\xi\|_{TV(\mathsf{S})} := \|\xi|_{\mathsf{S}}\|_{TV}$.

As mentioned previously, we will consider two problems which differ in their choice of the form of the stopping distribution q. In one problem we suppose that the distribution is a measure with density (with respect to Lebesgue) $q(x,t)$ on S (since $Leb(\overline{\mathsf{S}} \setminus \mathsf{S}) = 0$), and in the other we consider a more general measure q on $\overline{\mathsf{S}}$. In this problem we will specify that $\|q\|_{TV(\overline{\mathsf{S}})} < \infty$ or $\|\frac{q}{\rho}\|_{TV(\overline{\mathsf{S}})} < \infty$, and the latter condition ensures that q embeds no mass on $\overline{\mathsf{S}} \setminus \mathsf{S}$, so we can consider q as a measure on S instead.

With this in mind, define

$$\mathsf{X}(\lambda) := \Bigg\{ p : \bar{\mathsf{S}} \to \mathbb{R} \,\Big|\, \frac{p}{\bar{\rho}} \in L^\infty(\mathsf{S}),\ \frac{p(\cdot,0)}{\rho_0} \in L^\infty(\mathsf{T}),\ q = \frac{1}{2} p_{xx} - p_t \text{ exists weakly on } \mathsf{S},$$
$$\frac{q}{\bar{\rho}} \in L^\infty(\mathsf{S}),\ \text{and } \frac{\int_0^\infty p(\cdot,t)\mathrm{d}t}{L_\mu} \in L^\infty((x_*,x^*)) \Bigg\},$$
$$\widetilde{\mathsf{X}}(\lambda) := \Bigg\{ p : \bar{\mathsf{S}} \to \mathbb{R} \,\Big|\, \frac{p}{\bar{\rho}} \in L^\infty(\mathsf{S}),\ \frac{p(\cdot,0)}{\rho_0} \in L^\infty(\mathsf{T}),\ q = \frac{1}{2} p_{xx} - p_t \text{ is a measure on } \mathsf{S},$$
$$\left\| \frac{q}{\bar{\rho}} \right\|_{TV(\mathsf{S})} < \infty,\ \text{and } \frac{\int_0^\infty p(\cdot,t)\mathrm{d}t}{L_\mu} \in L^\infty((x_*,x^*)) \Bigg\},$$

equipped with the norms

$$\|p\|_{\mathsf{X}(\lambda)} := \left\| \frac{p}{\bar{\rho}} \right\|_{L^\infty(\mathsf{S})} + \left\| \frac{p(\cdot,0)}{\rho_0} \right\|_{L^\infty(\mathsf{T})} + \left\| \frac{q}{\bar{\rho}} \right\|_{L^\infty(\mathsf{S})} + \left\| \frac{\int_0^\infty p(\cdot,t)\mathrm{d}t}{L_\mu} \right\|_{L^\infty((x_*,x^*))},$$
$$\|p\|_{\widetilde{\mathsf{X}}(\lambda)} := \left\| \frac{p}{\bar{\rho}} \right\|_{L^\infty(\mathsf{S})} + \left\| \frac{p(\cdot,0)}{\rho_0} \right\|_{L^\infty(\mathsf{T})} + \left\| \frac{q}{\bar{\rho}} \right\|_{TV(\mathsf{S})} + \left\| \frac{\int_0^\infty p(\cdot,t)\mathrm{d}t}{L_\mu} \right\|_{L^\infty((x_*,x^*))}.$$

Here we have not specified a particular space for the measures q in $\widetilde{\mathsf{X}}(\lambda)$, however any finite Borel measure on a complete, separable metric space is regular, and therefore we have $q \in rca(\bar{\mathsf{S}})$.

We will also be interested in the unweighted spaces, which should correspond to $\mathsf{X}(0)$ and $\widetilde{\mathsf{X}}(0)$, however since the primal constraints $p(x,0) \leq \rho_0(x)$ and $\frac{1}{2}p_{xx}(x,t) - p_t(x,t) \geq 0$ ensure that $p(x,t) \leq \rho(x,t)$ for all (x,t), we choose to ignore the division by $\rho(x,t)$. Therefore, let

$$\mathsf{X} := \Bigg\{ p : \bar{\mathsf{S}} \to \mathbb{R} \,\Big|\, p \in L^\infty(\mathsf{S}),\ \frac{p(\cdot,0)}{\rho_0} \in L^\infty(\mathsf{T}),\ q = \frac{1}{2} p_{xx} - p_t \text{ exists weakly on } \mathsf{S}$$
$$q \in L^\infty(\mathsf{S}),\ \text{and } \frac{\int_0^\infty p(\cdot,t)\mathrm{d}t}{L_\mu} \in L^\infty((x_*,x^*)) \Bigg\},$$
$$\widetilde{\mathsf{X}} := \Bigg\{ p : \bar{\mathsf{S}} \to \mathbb{R} \,\Big|\, p \in L^\infty(\mathsf{S}),\ \frac{p(\cdot,0)}{\rho_0} \in L^\infty(\mathsf{T}),\ q = \frac{1}{2} p_{xx} - p_t \text{ is a measure on } \bar{\mathsf{S}},$$
$$\|q\|_{TV(\bar{\mathsf{S}})} < \infty,\ \text{and } \frac{\int_0^\infty p(\cdot,t)\mathrm{d}t}{L_\mu} \in L^\infty((x_*,x^*)) \Bigg\},$$

with the norms

$$\|p\|_{\mathsf{X}} := \|p\|_{L^\infty(\mathsf{S})} + \left\|\frac{p(\cdot,0)}{\rho_0}\right\|_{L^\infty(\mathsf{T})} + \|q\|_{L^\infty(\mathsf{S})} + \left\|\frac{w}{L_\mu}\right\|_{L^\infty((x_*,x^*))},$$
$$\|p\|_{\widetilde{\mathsf{X}}} := \|p\|_{L^\infty(\mathsf{S})} + \left\|\frac{p(\cdot,0)}{\rho_0}\right\|_{L^\infty(\mathsf{T})} + \|q\|_{TV(\overline{\mathsf{S}})} + \left\|\frac{w}{L_\mu}\right\|_{L^\infty((x_*,x^*))}.$$

Lemma 4.12. *Equipped with the above norms, the spaces* $\mathsf{X}, \tilde{\mathsf{X}}, \mathsf{X}(\lambda)$*, and* $\widetilde{\mathsf{X}}(\lambda)$ *are Banach spaces.*

Proof. First we note that all of the above define norms since $\|\cdot\|_{L^\infty}$ and $\|\cdot\|_{TV}$ are both norms on their respective spaces. It remains to show that the spaces are complete under these norms.

Take a Cauchy sequence $(p_n)_n$ in $\widetilde{\mathsf{X}}(\lambda)$, so the sequence $(\frac{p_n}{\bar\rho})_n$ is Cauchy in $L^\infty(\mathsf{S})$, a Banach space, and therefore converges to some $\hat p \in L^\infty(\mathsf{S})$. Let $p(x,t) = \bar\rho(x,t)\hat p(x,t)$, so $\frac{p}{\bar\rho} \in L^\infty(\mathsf{S})$, and

$$\|p_n - p\|_{L^\infty} \le \|\bar\rho\|_{L^\infty}\|\frac{p_n}{\bar\rho} - \hat p\|_{L^\infty} \to 0, \text{ as } n \to \infty,$$

since $\|\bar\rho\|_{L^\infty} < \infty$ as we assume that $\rho_0 \in L^\infty(\mathsf{T})$.

We have found a feasible limit point p such that $\frac{p}{\bar\rho} \in L^\infty(\mathsf{S})$, and we can extend this to $\overline{\mathsf{S}}$ by considering the limits of the $p_n(x,0)$, but we also need the correct conditions on $\frac{1}{2}p_{xx} - p_t$ and $w(x) := \int_0^\infty p(x,t)\mathrm{d}t = \int_0^\infty \bar\rho(x,t)\hat p(x,t)\mathrm{d}t$. Since $\hat p, \frac{\int_0^\infty \bar\rho(\cdot,t)\mathrm{d}t}{L_\mu^\cdot} \in L^\infty((x_*,x^*))$, for any $x \in (x_*,x^*)$ we have

$$\left|\frac{w(x)}{L_\mu^x}\right| \le \|\hat p\|_{L^\infty}\left|\frac{\int_0^\infty \bar\rho(x,t)\mathrm{d}t}{L_\mu^x}\right|,$$

so $\frac{w}{L_\mu^\cdot} \in L^\infty([x_*,x^*])$ as required.

Since $(p_n)_n$ is a sequence in $\widetilde{\mathsf{X}}(\lambda)$, for each n we have a corresponding stopping measure q_n such that $(\frac{q_n}{\bar\rho})_n$ is a Cauchy sequence in the Banach space of bounded measures (with the total variation norm), and therefore this sequence has a limit measure $\hat q$ with finite total variation. Define q to be the measure such that $q(A) = \int_A \bar\rho(x,t)\hat q(\mathrm{d}x,\mathrm{d}t)$. Then similarly to the above, $\|\frac{q}{\bar\rho}\|_{TV} < \infty$, and $\|q_n - q\|_{TV} \to 0$ as $n \to \infty$. All that remains to show is that $\frac{1}{2}p_{xx} - p_t = q$.

Using the definition of the distributional derivative, for any test function φ, we have

$$\langle \frac{1}{2}p_{xx} - p_t, \varphi \rangle = \langle p, \frac{1}{2}\varphi_{xx} + \varphi_t \rangle = \lim_{n\to\infty} \langle p_n, \frac{1}{2}\varphi_{xx} + \varphi_t \rangle = \lim_{n\to\infty} \langle q_n, \varphi \rangle = \langle q, \varphi \rangle,$$

where the second and final equalities hold since

$$\begin{aligned}\left| \langle p - p_n, \frac{1}{2}\varphi_{xx} + \varphi_t \rangle \right| &\leq \int |p(x,t) - p_n(x,t)| \left| \frac{1}{2}\varphi_{xx}(x,t) + \varphi_t(x,t) \right| \mathrm{d}x\mathrm{d}t \\ &\leq \|p - p_n\|_{L^\infty} \|\frac{1}{2}\varphi_{xx} + \varphi_t\|_{L^1} \\ &\to 0 \;\text{ as } n \to \infty \text{ for any } \varphi \in C_0^\infty(\mathsf{S}),\end{aligned}$$

and similarly

$$|\langle q - q_n, \varphi \rangle| \leq \|\varphi\|_{L^\infty} \|q - q_n\|_{TV} \to 0 \;\text{ for any } \varphi \in C_0^\infty(\mathsf{S}).$$

In the case of $\mathsf{X}(\lambda)$ note that we only have to restrict the above to the case where the q_n are functions such that $\frac{q_n}{\rho} \in L^\infty(\mathsf{S})$, in which case we have that $\langle \frac{1}{2}p_{xx} - p_t, \varphi \rangle = \langle q, \varphi \rangle$ for a function q such that $\frac{q}{\rho} \in L^\infty(\mathsf{S})$. The arguments are similar in the cases of X and $\widetilde{\mathsf{X}}$. □

Recall that our primal constraints are given by the bounded, linear operator A, where

$$Ap = \begin{pmatrix} \frac{1}{2}p_{xx} - p_t \\ (-p(x,0))_{x\in\mathsf{T}} \\ \left(-\int_0^\infty p(x,t)\mathrm{d}t\right)_{x\in[x_*,x^*]} \end{pmatrix}.$$

Then given a Y such that $A : \mathsf{X} \to \mathsf{Y}$, our dual variables are elements of Y^*, and similarly for $\mathsf{X}(\lambda)$, $\widetilde{\mathsf{X}}$, and $\widetilde{\mathsf{X}}(\lambda)$. The natural choices are as follows:

$$\begin{aligned}\mathsf{Y}(\lambda) &:= \mathsf{W}(\lambda) \times \mathsf{W}_0 \times \mathsf{Z}, \\ \widetilde{\mathsf{Y}}(\lambda) &:= \widetilde{\mathsf{W}}(\lambda) \times \mathsf{W}_0 \times \mathsf{Z}, \\ \mathsf{Y} &:= \mathsf{W} \times \mathsf{W}_0 \times \mathsf{Z}, \\ \widetilde{\mathsf{Y}} &:= \widetilde{\mathsf{W}} \times \mathsf{W}_0 \times \mathsf{Z},\end{aligned}$$

where

$$\begin{aligned}
\mathsf{W}(\lambda) &:= \left\{ w : \mathsf{S} \to \mathbb{R} \,\middle|\, \frac{w}{\bar{\rho}} \in L^\infty(\mathsf{S}) \right\}, \\
\widetilde{\mathsf{W}}(\lambda) &:= \left\{ w \in rca(\mathsf{S}) \,\middle|\, \left\| \frac{w}{\bar{\rho}} \right\|_{TV(\mathsf{S})} < \infty \right\}, \\
\mathsf{W} &:= L^\infty(\mathsf{S}), \\
\widetilde{\mathsf{W}} &:= rca\left(\overline{\mathsf{S}}\right), \\
\mathsf{W_0} &:= \left\{ w_0 : \mathsf{T} \to \mathbb{R} \,\middle|\, \frac{w_0}{\rho_0} \in L^\infty(\mathsf{T}) \right\}, \\
\mathsf{Z} &:= \left\{ z : (x_*, x^*) \to \mathbb{R} \,\middle|\, \frac{z}{\dot{L_\mu}} \in L^\infty((x_*, x^*)) \right\}.
\end{aligned}$$

Then we have the dual spaces

$$\begin{aligned}
\mathsf{Y}(\lambda)^* &:= \mathsf{W}(\lambda)^* \times \mathsf{W_0}^* \times \mathsf{Z}^*, \\
\widetilde{\mathsf{Y}}(\lambda)^* &:= \widetilde{\mathsf{W}}(\lambda)^* \times \mathsf{W_0}^* \times \mathsf{Z}^*, \\
\mathsf{Y}^* &:= \mathsf{W}^* \times \mathsf{W_0}^* \times \mathsf{Z}^*, \\
\widetilde{\mathsf{Y}}^* &:= \widetilde{\mathsf{W}}^* \times \mathsf{W_0}^* \times \mathsf{Z}^*,
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{W}(\lambda)^* &:= \left\{ \text{finitely additive measures } \eta \text{ on } \mathsf{S} \,\middle|\, \bar{\rho} \cdot \eta \in L^\infty(\mathsf{S})^* \right\}, \\
\widetilde{\mathsf{W}}(\lambda)^* &:= \left\{ \text{functionals } \eta \text{ on } \widetilde{\mathsf{W}}(\lambda) \,\middle|\, \bar{\rho} \cdot \eta \in rca(\overline{\mathsf{S}})^*, \text{where } (\bar{\rho} \cdot \eta)(w) = \eta(\bar{\rho} w)\ \forall w \in \widetilde{\mathsf{W}} \right\}, \\
\mathsf{W}^* &:= L^\infty(\mathsf{S})^*, \\
\widetilde{\mathsf{W}}^* &:= rca\left(\overline{\mathsf{S}}\right)^*, \\
\mathsf{W_0}^* &:= \left\{ \text{finitely additive measures } \eta_0 \text{ on } \mathsf{T} \,\middle|\, \rho_0 \cdot \eta_0 \in L^\infty(\mathsf{T})^* \right\}, \\
\mathsf{Z}^* &:= \left\{ \text{finitely additive measures } \nu \text{ on } (x_*, x^*) \,\middle|\, \dot{L_\mu} \cdot \nu \in L^\infty((x_*, x^*))^* \right\}.
\end{aligned}$$

We see that the dual variables η can be finitely additive measures or nets of functions, depending on the space we are working in. By Theorem 4.11, any $\eta \in rca(\mathsf{S})^*$ can be written as $(\eta_i)_{i\in\mathcal{I}}$ and we will write $\eta(x,t)$ to mean $(\eta_i(x,t))_{i\in\mathcal{I}}$. Each η_i defines a σ-finite measure $\hat{\eta}_i$ through $\hat{\eta}_i(A) = \int_A \eta_i(x,t)\mu_i(\mathrm{d}x, \mathrm{d}t)$. For this reason we will often generalise by saying that the η are measures, even if we have not specified which space we are working in. In this case we write $\eta(A)$ to be $(\hat{\eta}_i(A))_{i\in\mathcal{I}}$ for $A \subseteq \mathsf{S}$.

Note that any element of W^* defines a distribution since $C_0^\infty(\mathsf{S}) \subseteq \mathsf{W}$. Similarly, any $\varphi \in C_0^\infty(\mathsf{S})$ defines a measure $\nu_\varphi \in rca\left(\overline{\mathsf{S}}\right)$ by $\nu_\varphi(A) = \int_A \varphi(x,t)\mathrm{d}x\mathrm{d}t$ for $A \subseteq \overline{\mathsf{S}}$, so any $\eta \in \widetilde{\mathsf{W}}^*$ defines a distribution T_η such that $\langle T_\eta, \varphi\rangle = \eta(\nu_\varphi)$. This means we can also make sense of the distributional derivatives of these dual elements. Alternatively we will see later that we can consider any dual element η as the limit of continuous functions, and the distributional derivatives of these functions will also converge to some distribution which we could define to be the derivative of η.

To make our dual problem clearer, we will later consider measures $\tilde{\eta}$ on S such that for any $A \subseteq \mathsf{S}$, $\tilde{\eta}(A) = \eta(A \cap \mathsf{S}) + \eta_0(A \cap (\mathsf{T} \times \{0\}))$ for $\eta \in \mathsf{W}^*$ (or $\widetilde{\mathsf{W}}^*, \mathsf{W}(\lambda)^*, \widetilde{\mathsf{W}}(\lambda)^*$) and $\eta_0 \in \mathsf{W_0}^*$. We will therefore write $(\tilde{\eta}, \nu) \in \mathsf{Y}^*$ to mean that $(\eta, \eta_0, \nu) \in \mathsf{Y}^*$ and that $\tilde{\eta}$ is defined by η and η_0 as above. In the case where η and η_0 are functions, this can be seen as specifying that $\eta(x,0) = \eta_0(x)$. To persuade the reader that this is a natural definition, consider the problem $\tilde{\mathsf{X}}$ and take $p \in \tilde{\mathsf{X}}$ with corresponding stopping density q. For $t > 0$, $\eta(x,t)$ is the dual variable corresponding to the primal constraint $q := (\frac{1}{2}p_{xx} - p_t) \geq 0$. Similarly, $\eta_0(x)$ corresponds to the condition $p(x,0) \leq \rho_0(x)$, which can be rewritten as $p(x,0) + q(x,0) = \rho_0(x)$ *and* $q(x,0) \geq 0$. Therefore, both η and η_0 are dual variables to constraints specifying that the stopping density is non-negative, and therefore η_0 should correspond to the value of η on $\mathsf{T} \times \{0\}$.

4.3 Primal Problem and Attainment

We will consider the example of optimising the payoff of the European call option on an LETF, which we know has an optimal K-cave barrier solution. All results should be adaptable to certain examples of Root, Rost, and cave embedding solutions, but some arguments will be dependent on the payoff function, and we therefore present the K-cave case and provide comments on how to consider the other problems.

Our primal problem, optimising over the space X, is

$$\begin{aligned}\sup_p \Phi(p) &:= \left\langle F, \frac{1}{2}p_{xx} - p_t \right\rangle + \int F(x,0)\left(\rho_0(x) - p(x,0)\right)\\ &= \left\langle \frac{1}{2}F_{xx} + F_t, p \right\rangle + \int F(x,0)\rho_0(x)\mathrm{d}x\end{aligned}$$

over functions $p(x,t)$ subject to

- $p \in \mathsf{X}$
- $p(x,t) \geq 0, \qquad \forall (x,t) \in \overline{\mathsf{S}}$
- $\frac{1}{2}p_{xx} - p_t \geq_{w(\mathsf{S})} 0,$
- $\int_0^\infty p(x,t)\mathrm{d}t \leq L_\mu^x \qquad \forall x \in [x_*, x^*]$
- $p(x,0) \leq \rho_0(x) = \bar{\rho}(x,0), \qquad \forall x \in \mathsf{T},$

where $F(x,t) = (h(x,t) - k)_+$, $h(x,t) = \mathrm{e}^{\beta x - \frac{1}{2}\beta^2 t}$ and $k, \beta > 0$ are constants.

Let $\mathcal{P}$ be the set of primal-feasible functions $p(x,t)$, and denote the optimal value by $\mathsf{p} := \sup_{p \in \mathcal{P}} \Phi(p)$. We can define similarly the primal problem for our spaces $\tilde{\mathsf{X}}$, $\mathsf{X}(\lambda)$, and $\tilde{\mathsf{X}}(\lambda)$, and their corresponding $\tilde{\mathcal{P}}, \mathcal{P}(\lambda), \tilde{\mathcal{P}}(\lambda)$ and $\tilde{\mathsf{p}}, \mathsf{p}(\lambda), \tilde{\mathsf{p}}(\lambda)$.

Recall that $K(x) := \frac{2x}{\beta} - \frac{2}{\beta^2}\ln(k)$ and let $x_0 = \frac{1}{\beta}\ln(k)$ so $K(x_0) = 0$. Then $\frac{1}{2}F_{xx} + F_t$ is the distribution on S such that for any $\varphi \in C_0^\infty(\mathsf{S})$,

$$\begin{aligned}
\left\langle \frac{1}{2}F_{xx} + F_t, \varphi \right\rangle &= \left\langle F, \frac{1}{2}\varphi_{xx} - \varphi_t \right\rangle \\
&= \int_{x_*}^{x^*} \int_0^\infty F(x,t) \left(\frac{1}{2}\varphi_{xx}(x,t) - \varphi_t(x,t) \right) \mathrm{d}t\mathrm{d}x \\
&= \int_{x_0}^{x^*} \int_0^{K(x)} F(x,t) \left(\frac{1}{2}\varphi_{xx}(x,t) - \varphi_t(x,t) \right) \mathrm{d}t\mathrm{d}x \\
&= \int_{x_0}^{x^*} \int_0^{K(x)} \varphi(x,t) \left(\frac{1}{2}h_{xx}(x,t) + h_t(x,t) \right) \mathrm{d}t\mathrm{d}x \\
&\qquad\qquad + \int_{x_0}^{x^*} \varphi(x,K(x))h(x,K(x))\mathrm{d}x \\
&= k\int_{x_0}^{x^*} \varphi(x,K(x))\mathrm{d}x.
\end{aligned}$$

In particular, this maps any non-negative test function to a non-negative real, so $\frac{1}{2}F_{xx} + F_t \geq_{w(\mathsf{S})} 0$, and $F(W_t,t)$ is a submartingale. To maximise $\mathbb{E}\left[F(W_\tau,\tau)\right]$ over stopping times τ, we then want to let W run for as long as possible before stopping. In our current setup we wish to maximise $\Phi(p) = \left\langle \frac{1}{2}F_{xx} + F_t, p \right\rangle$ over $p \in \mathcal{P}$, and so if $\frac{1}{2}F_{xx} + F_t \geq_{w(\mathsf{S})} 0$ then we want to ensure that p is large for as long as possible. Any optimal p should then have maximal local time, i.e. $\int_0^\infty p(x,t)\mathrm{d}t = L_\mu^x$ for all $x \in [x_*, x^*]$. If we also have $p(x,0) = \rho_0(x)$, then the stopped process corresponding to p is a Brownian motion W with $W_0 \sim \nu$ and $W_\tau \sim \mu$. We prove this in the next lemma using the ideas of

Lemma 2.8.

Lemma 4.13. *Provided the feasible spaces are non-empty,*

$$\tilde{\mathsf{p}} := \sup_{p\in\tilde{\mathcal{P}}} \Phi(p) = \sup_{\substack{p\in\tilde{\mathcal{P}} \\ \int_0^\infty p(x,t)\mathrm{d}t = L_\mu^x\ \forall x \\ p(x,0)=\rho_0(x)\forall x}} \Phi(p) = \sup_{\substack{p\in\tilde{\mathcal{P}} \\ p(x,0)=\rho_0(x)\ \forall x}} \Phi(p),$$

and similarly for $\mathsf{p}, \mathsf{p}(\lambda), \tilde{\mathsf{p}}(\lambda)$.

Proof. Suppose we have $p \in \tilde{\mathcal{P}}$ such that $p(x,0) < \rho_0(x)$ for some $x \in \mathsf{T}$. We consider allowing the extra $\rho_0(x) - p(x,0)$ mass to run for some small length of time, improving our payoff. This increases the local time of the process, and we may already have equality in the local time conditions, meaning that the resulting stopping rule is not feasible. It will however be feasible for a problem that allows slightly more local time, so $\int_0^\infty p(x,t)\mathrm{d}t \le L^x_{\mu_\varepsilon}$ for some μ_ε.

To be precise, fix some small $\varepsilon > 0$, and take any $p \in \tilde{\mathcal{P}}$ ($\mathcal{P}$ works similarly). The function p defines a randomised stopping time, τ_p, for a Brownian motion. For each path of the Brownian motion, simulate an independent uniform random variable U on $(0,1)$. For $x \in (x_*, x^*)$, let $\mathcal{I}_x := (x - \frac{1}{2}, x + \frac{1}{2}) \cap (\frac{1}{2}(x^* + x), \frac{1}{2}(x_* + x))$. If, according to τ_p, a path is stopped in a neighbourhood of (x,t), for $x \in (x_*, x^*)$, allow the path to continue running until it exits the box

$$\left\{ (y,s) \in \overline{\mathsf{S}} :\ |x-y| < U,\ s - t < U \frac{\varepsilon}{1-\varepsilon} \inf_{z\in\mathcal{I}_x} L^z_\mu \right\}.$$

Note that for $p \in \tilde{\mathcal{P}}$ this mechanism smooths out the stopping measure (or the mass that is stopped within S) to ensure it is absolutely continuous with respect to Lebesgue and therefore has a density. We use this approach in Section 4.5.1. The above method does not smooth out the mass stopped at x_* or x^*.

Allowing the stopped mass at time 0 to evolve in this way, we attain some $\bar{p}$ such that $\bar{p}(x,0) = \rho_0(x)$. Importantly, the choice of the smoothing ensures that we can find some $L^x_{\mu_\varepsilon}$ such that for *any* $p \in \tilde{\mathcal{P}}$, the resulting $\bar{p}$ is such that $\int_0^\infty \bar{p}(x,t)\mathrm{d}t \le L^x_{\mu_\varepsilon}$ for all $x \in [x_*, x^*]$. Also, for any $x \in (x_*, x^*)$ we have

$$\begin{aligned}
L^x_{\mu_\varepsilon} - L^x_\mu &\leq \frac{\varepsilon}{1-\varepsilon} \int_{x-1}^{x+1} \inf_{z \in (y-\frac{1}{2}, y+\frac{1}{2})} L^z_\mu \mathrm{d}y \\
&\leq \frac{\varepsilon}{1-\varepsilon} L^x_\mu \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \mathrm{d}y \\
&\leq \frac{\varepsilon}{1-\varepsilon} L^x_\mu,
\end{aligned}$$

so

$$(1-\varepsilon) L^x_{\mu_\varepsilon} \leq L^x_\mu. \tag{4.1}$$

Consider the primal problem $\tilde{\mathcal{P}}^\varepsilon$ which is identical to $\tilde{\mathcal{P}}$ except that we use L_{μ_ε} in place of L_μ. Similarly we can define $\mathcal{P}^\varepsilon$ and $\tilde{\mathsf{p}}^\varepsilon, \mathsf{p}^\varepsilon$. Then, for any $p \in \tilde{\mathcal{P}}$, the above protocol gives a feasible $\bar{p} \in \tilde{\mathcal{P}}^\varepsilon$ such that $\Phi(\bar{p}) \geq \Phi(p)$ and $\bar{p}(x,0) = \rho_0(x)$ for all $x \in \mathsf{T}$. Clearly then

$$\sup_{\substack{p \in \tilde{\mathcal{P}} \\ p(x,0)=\rho_0(x)\ \forall x}} \Phi(p) \leq \tilde{\mathsf{p}} \leq \lim_{\varepsilon \to 0} \sup_{\substack{p \in \tilde{\mathcal{P}}^\varepsilon \\ p(x,0)=\rho_0(x)\ \forall x}} \Phi(p) \leq \lim_{\varepsilon \to 0} \tilde{\mathsf{p}}^\varepsilon.$$

Now take any $p \in \tilde{\mathcal{P}}^\varepsilon$, with corresponding stopping time τ_p, and for $x \in [x_*, x^*]$ let $T(x) := \inf\{t \geq 0 : \int_0^\infty p(x,t)\mathrm{d}x = L^x_\mu\}$ (where $\inf \varnothing = \infty$). Stopping a Brownian motion, W, according to $\tau_p \wedge \inf\{t \geq 0 : t \geq T(W_t)\}$ gives some $\bar{p} \in \tilde{\mathcal{P}}$ with $\bar{p}(x,0) = p(x,0)$ and $\int_0^\infty \bar{p}(x,t)\mathrm{d}t \leq L^x_\mu$ for all x. For $\mathcal{P}$ we choose a smooth equivalent of this stopping rule. Also,

$$\begin{aligned}
\Phi(\bar{p}) &\geq \Phi(p) - k \int_{x_*}^{x^*} \left(L^x_{\mu_\varepsilon} - L^x_\mu \right) \mathrm{d}x \\
&\geq \Phi(p) - k \frac{\varepsilon}{1-\varepsilon} \int_{x_*}^{x^*} L^x_\mu \mathrm{d}x \\
&\to \Phi(p) \quad \text{as } \varepsilon \to 0,
\end{aligned}$$

since $\int_{x_*}^{x^*} L^x_\mu \mathrm{d}x < \infty$. It follows that

$$\sup_{\substack{p \in \tilde{\mathcal{P}} \\ p(x,0)=\rho_0(x)\ \forall x}} \Phi(p) \geq \lim_{\varepsilon \to 0} \sup_{\substack{p \in \tilde{\mathcal{P}}^\varepsilon \\ p(x,0)=\rho_0(x)\ \forall x}} \Phi(p),$$

and so

$$\sup_{\substack{p \in \tilde{\mathcal{P}} \\ p(x,0)=\rho_0(x)\ \forall x}} \Phi(p) = \tilde{\mathsf{p}}.$$

Now take any $p \in \tilde{\mathcal{P}}$ (the argument for $\mathcal{P}$ is almost identical) such that $p(x,0) = \rho_0(x)$ but $\int_0^\infty p(x,t)dt = L_\mu^x - \varepsilon$ for some $x \in (x_*, x^*)$ and $\varepsilon > 0$, where without loss of generality there exists $s > 0$ such that $q(A_{x,s}) > 0$ for some $A_{x,s}$ a small neighbourhood of (x,s), i.e. we stop some mass at (x,s). Note that $\int_0^\infty p(\cdot,t)dt$ is continuous (from the local time interpretation or results on continuity of solutions of the heat equation, c.f. Nash [1958]), so there exists $\delta > 0$ such that $\int_0^\infty p(y,t)\mathrm{d}t < L_\mu^y - \frac{\varepsilon}{2}$ for all $y \in (x-\delta, x+\delta)$.

Let $\kappa := \frac{\varepsilon}{2} \wedge q(A_{x,s})$ and consider letting κ more paths run from the point (x,s) which we then stop uniformly in $(x-\delta, x+\delta)$ (any smooth stopping within this region will also work for $\mathcal{P}$, though the mass should be stopped within some small time for the arguments of $\mathcal{P}(\lambda)$ and $\tilde{\mathcal{P}}(\lambda)$). This new stopping rule defines some $\bar{p} \in \tilde{\mathcal{P}}$ with $\bar{p}(x,0) = p(x,0)$ and $\bar{p}(x,t) \geq p(x,t)$, so clearly $\Phi(\bar{p}) \geq \Phi(p)$. We can therefore, without loss of generality, consider optimising over

$$\left\{ p \in \tilde{\mathcal{P}} \middle| \int_0^\infty p(x,t)\mathrm{d}t = L_\mu^x \,\forall x \in [x_*, x^*],\ p(x,0) = \rho_0(x) \,\forall x \in \mathsf{T} \right\},$$

provided this space is non-empty, so

$$\tilde{\mathsf{p}} := \sup_{p \in \tilde{\mathcal{P}}} \Phi(p) = \sup_{\substack{p \in \tilde{\mathcal{P}} \\ \int_0^\infty p(x,t)\mathrm{d}t = L_\mu^x \,\forall x \\ p(x,0) = \rho_0(x) \forall x}} \Phi(p).$$

$\square$

From the form of our payoff, we see that when we release mass in the arguments of Lemma 4.13, we may not have a strictly greater value of the objective function, since for example we have that $F(x,t) = 0$ for all x and all sufficiently large t. If $F(W_t, t)$ was a strict submartingale we would conclude that any optimiser p^* *must* have $p^*(x,0) = \rho_0(x)$ and $\int_0^\infty p(x,t)\mathrm{d}t = L_\mu^x$ for all suitable x. In the case of the LETF payoff we can choose to run the above argument even in the constant region of the payoff, and this flexibility corresponds to ideas discussed in Section 3.6 where we introduce the idea of a regular K-cave barrier.

When transferring some later arguments to the cases of the Root, Rost, and cave embeddings, it will be important that the payoffs are increasing functions of time, and to do this we repeat the ideas of Section 3.7.

We can now show again that there is a solution to (OptSEP) for this payoff.

Lemma 4.14. *The primal value* $\tilde{\mathsf{p}}$ *is attained, and in particular for a Brownian motion* W *with* $W_0 \sim \nu$*, there is a randomised stopping time* τ^* *which maximises*

$$\mathbb{E}\left[F(W_\tau, \tau)\right]$$

over all randomised stopping times such that $W_\tau \sim \mu$*.*

Proof. Consider the set

$$\mathsf{P}(\tilde{\mathsf{X}}) := \left\{ p \in \tilde{\mathcal{P}} \middle| \, p(x,0) = \rho_0(x) \text{ and } \int_0^\infty p(x,t)\mathrm{d}t = L_\mu^x, \text{ Leb-a.e. in } [x_*, x^*] \right\}.$$

Any $p \in \mathsf{P}(\tilde{\mathsf{X}}) \subseteq \tilde{\mathcal{P}}$ defines a measure, σ_p, by $\sigma_p(A) := \int_A p(x,t)\mathrm{d}x\mathrm{d}t$ for $A \subseteq \overline{\mathsf{S}}$. From the properties of p we see that this measure defines the evolution of a stopped Brownian motion W with $W_0 \sim \nu$, where $\sigma_p(A) = \int_0^\infty \mathbb{P}\left((W_t, t) \in A, t < \tau\right)\mathrm{d}t$ for some stopping time τ. By the local time condition we know that $W_\tau \sim \mu$.

Recall that $L^\infty(\mathsf{S})$ is defined up to equivalence classes of Leb-a.e. equivalent p, and so the mapping from p to σ_p is one-to-one. Also, any such measure σ is dominated by the usual Gaussian transition density for Brownian motion, and so by the Radon-Nikodym theorem there exists a function p such that $\sigma \equiv \sigma_p$. We can easily check that $p \in \mathsf{P}(\tilde{\mathsf{X}})$, so we have a bijection between $\mathsf{P}(\tilde{\mathsf{X}})$ and this set of measures. In particular, $\mathsf{P}(\tilde{\mathsf{X}}) \neq \varnothing$.

For square-integrable μ,

$$\sigma_p\left(\overline{\mathsf{S}}\right) = \int_{x_*}^{x^*} \int_0^\infty p(x,t)\mathrm{d}t\mathrm{d}x = \int_{x_*}^{x^*} L_\mu^x \mathrm{d}x = \int_{x_*}^{x^*} x^2 \mu(\mathrm{d}x) < \infty,$$

so $\|\sigma_p\|_{TV} = \int_{x_*}^{x^*} L_\mu^x \mathrm{d}x$ for any $p \in \mathsf{P}(\tilde{\mathsf{X}})$. Note also that for any $p \in \mathsf{P}(\tilde{\mathsf{X}})$, $p(x,t) \leq \rho(x,t)$ for any $(x,t) \in \overline{\mathsf{S}}$. In particular, for any $\varepsilon > 0$, there is a large T such that

$$\sigma_p\left([x_*, x^*] \times [T, \infty)\right) \leq \int_{x_*}^{x^*} \int_T^\infty \rho(x,t)\mathrm{d}t\mathrm{d}x = \mathbb{P}\left(H_{x^*}(W) \wedge H_{x^*}(W) > T\right) < \varepsilon.$$

The set of measures induced by elements of $\mathsf{P}(\tilde{\mathsf{X}})$ is therefore tight, and so its closure is sequentially compact by Prokhorov's theorem, so there is a sequence $(p_n)_n \in \mathsf{P}(\tilde{\mathsf{X}})$ and a measure σ such that $\Phi(p_n) \to \sup_{p \in \mathsf{P}(\tilde{\mathsf{X}})} \Phi(p)$ and $\sigma_{p_n} \to \sigma$ weakly as $n \to \infty$. We show that our set of measures is closed, so σ is defined by some $\bar{p} \in \mathsf{P}(\tilde{\mathsf{X}})$.

Since all the measures are non-negative,

$$\|\sigma\|_{TV} = \sigma\left(\overline{\mathsf{S}}\right) = \lim_{n\to\infty}\int_{\overline{\mathsf{S}}} p_n(x,t)\mathrm{d}x\mathrm{d}t = \lim_{n\to\infty}\|\sigma_{p_n}\|_{TV} = \int_{x_*}^{x^*} L_\mu^x \mathrm{d}x,$$

and furthermore there is a subsequence $(p_{n_k})_k$ such that $\|\sigma - \sigma_{p_{n_k}}\|_{TV} \to 0$ as $k\to\infty$. Then $\sigma \in rca\left(\overline{\mathsf{S}}\right)$ since $\sigma_{p_{n_k}} \in rca\left(\overline{\mathsf{S}}\right)$ for each k, and this is complete under the total variation norm.

Fix any $A \subseteq \overline{\mathsf{S}}$ such that $\mathrm{Leb}(A) = 0$, so that $\sigma_{p_{n_k}}(A) = 0$ for any k, and take an increasing sequence of open, measurable sets $(F_j)_j$, and a decreasing sequence of closed, measurable sets $(G_j)_j$ such that $F_j \subseteq A \subseteq G_j$ for any j. The Portmanteau Theorem and the regularity of σ imply that for any j,

$$\inf_k \sigma_{p_{n_k}}(F_j) \le \limsup_{k\to\infty}\sigma_{p_{n_k}}(F_j) \le \sigma(A) \le \liminf_{k\to\infty}\sigma_{p_{n_k}}(G_j) \le \sup_k \sigma_{p_{n_k}}(G_j).$$

Since the $\sigma_{p_{n_k}}$ are regular, $\lim_{j\to\infty}\sigma_{p_{n_k}}(F_j) = \sigma_{p_{n_k}}(A) = 0$ for every k, and so $\lim_{j\to\infty}\inf_k \sigma_{p_{n_k}}(F_j) = 0$ also. The upper bound works similarly and we have $\sigma(A) = 0$, so $\sigma \ll \mathrm{Leb}$. There exists a function $\bar{p}$ on $\overline{\mathsf{S}}$ with $\|\bar{p}\|_{L^1(\overline{\mathsf{S}})} = \int_{x_*}^{x^*} L_\mu^x \mathrm{d}x$ such that $\sigma(A) = \int_A \bar{p}(x,t)\mathrm{d}x\mathrm{d}t$ for any A. It remains to show that $\bar{p} \in \mathsf{P}(\tilde{\mathsf{X}})$.

Note that since $\sigma \ll \mathrm{Leb}$, any set $A = (x_1,x_2)\times(t_1,t_2)$ is a continuity set of σ. Then the Portmanteau Theorem implies that $\int_{x_1}^{x_2}\int_0^\infty \bar{p}(x,t)\mathrm{d}x\mathrm{d}t = \int_{x_1}^{x_2} L_\mu^x \mathrm{d}x$ for any $x_1 < x_2$ and so we must have $\int_0^\infty \bar{p}(x,t)\mathrm{d}t = L_\mu^x$ Leb-a.e. Similarly, for $x_1 < x_2$,

$$\int_{x_1}^{x_2} \bar{p}(x,0)\mathrm{d}x = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon\int_{x_1}^{x_2}\bar{p}(x,t)\mathrm{d}x\mathrm{d}t = \lim_{\varepsilon\to 0}\lim_{k\to\infty}\frac{1}{\varepsilon}\int_0^\varepsilon\int_{x_1}^{x_2} p_{n_k}(x,t)\mathrm{d}x\mathrm{d}t.$$

Also

$$\int_{x_1}^{x_2} p_{n_k}(x,\varepsilon)\mathrm{d}x \le \frac{1}{\varepsilon}\int_0^\varepsilon\int_{x_1}^{x_2} p_{n_k}(x,t)\mathrm{d}x\mathrm{d}t \le \int_{x_1}^{x_2}\rho_0(x)\mathrm{d}x,$$

and for any $\delta > 0$ we can find $\varepsilon > 0$ such that for sufficiently large k, $\int_{x_1}^{x_2} p_{n_k}(x,\varepsilon)\mathrm{d}x \ge \int_{x_1}^{x_2}\rho_0(x)\mathrm{d}x - \delta$, and so $\int_{x_1}^{x_2}\bar{p}(x,0)\mathrm{d}x = \int_{x_1}^{x_2}\rho_0(x)\mathrm{d}x$.

It remains to show that $\frac{1}{2}\bar{p}_{xx} - \bar{p}_t$ is a measure with finite total variation so that $\bar{p} \in \tilde{\mathsf{X}}$. For any $\varphi \in C_0^\infty(\mathsf{S})$ the function $\frac{1}{2}\varphi_{xx} + \varphi_t$ is continuous and bounded so by the weak convergence of the p_{n_k},

$$\begin{aligned}\int \varphi(x,t) q_{n_k}(\mathrm{d}x, \mathrm{d}t) &= \int \left(\frac{1}{2}\varphi_{xx} + \varphi_t\right)(x,t) p_{n_k}(x,t) \mathrm{d}x\mathrm{d}t \\ &\to \int \left(\frac{1}{2}\varphi_{xx} + \varphi_t\right)(x,t) \bar{p}(x,t) \mathrm{d}x\mathrm{d}t \\ &= \langle \phi, \frac{1}{2}\bar{p}_{xx} - \bar{p}_t \rangle,\end{aligned}$$

meaning the distributional derivatives q_{n_k} converge weakly to the stopping distribution of σ. Furthermore, since $\|q_n\|_{TV} = 1$ for each n, we can without loss of generality choose our subsequence so that the measures q_{n_k} converge in total variation to a measure in $rca\left(\overline{\mathsf{S}}\right)$, and this limit must agree with the weak limit. Then $\bar{q} = \frac{1}{2}\bar{p}_{xx} - \bar{p}_t$ is a measure with total variation 1, and therefore $\bar{p} \in \tilde{\mathsf{X}}$.

The above $\bar{p} \in \mathsf{P}(\tilde{\mathsf{X}})$ is such that

$$\Phi(\bar{p}) = \sup_{p \in \mathsf{P}(\tilde{\mathsf{X}})} \Phi(p).$$

By Lemma 4.13, $\bar{p}$ is also an optimiser over $\tilde{\mathcal{P}}$, so $\tilde{\mathsf{p}}$ is attained by $\bar{p}$.

$\square$

4.4 Strong Duality in Weighted Space

Recall that our problem is of the form

$$\sup_p \Phi(p) \quad \text{over } p \in \mathsf{X}$$

subject to

- $p \geq 0$
- $Ap \geq B$,

where Φ is linear, and A, B are given by

$$Ap = \begin{pmatrix} \frac{1}{2}p_{xx} - p_t \\ (-p(x,0))_{x \in \mathsf{T}} \\ \left(-\int_0^\infty p(x,t)\mathrm{d}t\right)_{x \in [x_*, x^*]} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ (-\rho_0(x))_{x \in \mathsf{T}} \\ \left(-L_\mu^x\right)_{x \in [x_*, x^*]} \end{pmatrix}.$$

We wish to prove duality using Theorem 2.4, from Borwein and Zhu [2006, Theorem 4.4.3], which we restate here for ease of reading.

Theorem 4.15. *Let* X *and* Y *be Banach spaces, let* $f : \mathsf{X} \to \mathbb{R} \cup \{\infty\}$ *and* $g : \mathsf{Y} \to \mathbb{R} \cup \{\infty\}$ *be convex functions and let* $A : \mathsf{X} \to \mathsf{Y}$ *be a bounded linear map. Define the primal and dual values* $\mathsf{p}, \mathsf{d} \in [-\infty, \infty]$ *by the Fenchel problems*

$$\begin{aligned} \mathsf{p} &= \inf_{x \in \mathsf{X}} \{f(x) + g(Ax)\} \\ \mathsf{d} &= \sup_{y^* \in \mathsf{Y}^*} \{-f^*(A^* y^*) - g^*(-y^*)\}. \end{aligned}$$

Then $\mathsf{p} = \mathsf{d}$*, and the supremum in the dual problem is achieved if either of the following hold*

(i) $0 \in \operatorname{core}(\operatorname{dom}(g) - A\operatorname{dom}(f))$ *and* f, g *are lower semi-continuous*

(ii) $A\operatorname{dom}(f) \cap \operatorname{cont}(g) \neq \varnothing$.

To write our problem in this form we choose

$$\begin{aligned} f(x) &= \begin{cases} -\Phi(p), & p \geq 0 \, \forall (x,t), \; p(x,0) \leq \rho_0(x) \, \forall x \\ \infty, & \text{otherwise}, , \end{cases} \\ g((w, w_0, z)) &= \begin{cases} 0, & (w, w_0, z) \geq B^T \\ \infty, & \text{otherwise}, \end{cases} \end{aligned}$$

so

$$\begin{aligned} f^*(p^*) &= \begin{cases} \int F(x,0)\rho_0(x)\mathrm{d}x, & p^* \leq_w -(\frac{1}{2}F_{xx} + F_t) \\ \infty, & \text{otherwise}, \end{cases} \\ g^*((w^*, w_0^*, z^*)) &= \begin{cases} -\int L_\mu^x z^*(\mathrm{d}x) - \int_\infty^\infty \rho_0(x) w_0^*(\mathrm{d}x), & w^* \leq_w 0, w_0^* \leq_w 0, z^* \leq_w 0 \\ \infty, & \text{otherwise}. \end{cases} \end{aligned}$$

Also, A^* is the map $A^* : \mathsf{Y}^* \to \mathsf{X}^*$ such that $\langle (w^*, w_0^*, z^*), Ap \rangle = \langle A^*(w^*, w_0^*, z^*), p \rangle$ for any $(w^*, w_0^*, z^*) \in \mathsf{Y}^*$ and $p \in \mathsf{X}$. We have

$$\begin{aligned} \langle (w^*, w_0^*, z^*), Ap \rangle &= \left\langle \frac{1}{2} p_{xx} - p_t, w^* \right\rangle - \int_{x_*}^{x^*} p(x,0) w_0^*(\mathrm{d}x) - \int_{x_*}^{x^*} \int_0^\infty p(x,t)\mathrm{d}t z^*(\mathrm{d}x) \\ &= \left\langle \frac{1}{2} \tilde{w}_{xx}^* + \tilde{w}_t^* - \tilde{z}^*, p \right\rangle \end{aligned}$$

so $A^*(w^*, w_0^*, z^*) = \frac{1}{2}\tilde{w}^*_{xx} + \tilde{w}^*_t - \tilde{z}^*$, where $\tilde{z}^*(D) = \int_D z^*(\mathrm{d}x)\mathrm{d}t$ and $\tilde{w}^*(D) = w^*(D \cap \mathsf{S}) + w_0^*(D \cap (\mathsf{T} \times \{0\}))$ for $D \subseteq \mathsf{S}$.

There is a boundary term at $t = 0$ from $\langle \frac{1}{2}p_{xx} - p_t, w^* \rangle$, but by defining $\tilde{w}$ as above we ensure that this term cancels with $\int_{x_*}^{x^*} p(x, 0) w_0^*(\mathrm{d}x)$. We can then think of $\tilde{w}^*$ as 'starting' as w_0^* at time 0 and then evolving according to $\frac{1}{2}\tilde{w}_{xx} + \tilde{w}_t$. This choice makes much clearer the relationship between w^* and w_0^*.

We can make sense of terms such as $\left\langle \frac{1}{2}\tilde{w}^*_{xx} + \tilde{w}^*_t, p \right\rangle$ since $\frac{1}{2}\tilde{w}^*_{xx} + \tilde{w}^*_t$ defines a distribution for any w^*, and any $p \in \mathsf{X} \subseteq L^1(\mathsf{S})$ is the limit of test functions in $(\varphi_n)_n \in C_0^\infty(\mathsf{S})$, so we can define

$$\left\langle \frac{1}{2}\tilde{w}^*_{xx} + \tilde{w}^*_t, p \right\rangle = \lim_{n \to \infty} \left\langle \frac{1}{2}\tilde{w}^*_{xx} + \tilde{w}^*_t, \varphi_n \right\rangle.$$

Then, our dual problem is

$$\inf_{\eta, \nu} \Psi(\eta, \nu) := \int L^x_\mu \nu(\mathrm{d}x) + \int \rho_0(x) F(x, 0) \mathrm{d}x + \int \rho_0(x) \eta_0(\mathrm{d}x)$$

over measures η on $\overline{\mathsf{S}}$, and ν on (x_*, x^*), subject to

- $(\eta, \nu) \in \mathsf{Y}^*$
- $\eta \geq_{w(\mathsf{S})} 0, \nu \geq_{w((x_*, x^*))} 0$ (D1)
- $\frac{1}{2}\eta_{xx} + \eta_t - \tilde{\nu} \leq_{w(\mathsf{S})} -\left(\frac{1}{2}F_{xx} + F_t\right)$ (D2)

where $\tilde{\nu}^*(\mathrm{d}x, \mathrm{d}t) = \nu^*(\mathrm{d}x)\mathrm{d}t$ and $\eta_0(\mathrm{d}x) = \eta(\mathrm{d}x \times \{0\})$.

Denote the dual feasible spaces by $\mathcal{D}$, $\tilde{\mathcal{D}}$, $\mathcal{D}(\lambda)$, and $\tilde{\mathcal{D}}(\lambda)$, and the optimal values by d, $\tilde{\mathsf{d}}$, $\mathsf{d}(\lambda)$, and $\tilde{\mathsf{d}}(\lambda)$. We prove the following duality result using condition (i) of Theorem 2.4.

Theorem 4.16. *We have strong duality with the space $\mathsf{X}(\lambda)$ in the sense that*

$$\mathsf{p}(\lambda) = \mathsf{d}(\lambda),$$

and the optimal dual value $\mathsf{d}(\lambda)$ is attained.

Proof. First note that the f and g we provide above are lower semi-continuous and give the correct primal problem. Condition (i) can be rewritten

$$0 \in \operatorname{core}(\operatorname{dom}(g) - A\operatorname{dom}(f)) \Longleftrightarrow \bigcup_{\alpha>0} \alpha(\operatorname{dom}(g) - A\operatorname{dom}(f)) = \mathsf{Y}(\lambda).$$

It is then sufficient to show that for any $(w(x,t), w_0(x), z(x)) \in \mathsf{Y}(\lambda)$, there exists $\alpha > 0$, $p(x,t) \in \mathsf{X}(\lambda)$ and $(y(x,t), y_0(x), \bar{y}(x)) \in \mathsf{Y}(\lambda)$ such that

$$\begin{array}{llll} w(x,t) = \alpha\left(y(x,t) - \frac{1}{2}p_{xx}(x,t) + p_t(x,t)\right) & \forall(x,t), & p(x,t) \geq 0,\ y(x,t) \geq 0 & \forall(x,t) \\ w_0(x) = \alpha\left(y_0(x) + p(x,0)\right) & \forall x, & y_0(x) \geq -\rho_0(x) & \forall x \\ z(x) = \alpha\left(\bar{y}(x) + \int_0^\infty p(x,t)\mathrm{d}t\right) & \forall x, & \bar{y}(x) \geq -L_\mu^x & \forall x. \end{array}$$

Take any $(w,z) \in \mathsf{Y}$ and define the function p to be the solution of

$$\frac{1}{2}p_{xx}(x,t) - p_t(x,t) = \frac{1}{\alpha}\left(|w(x,t)| + \bar{\rho}(x,t)\right), \quad p(x,0) = \frac{1}{\alpha}\rho_0(x),$$

for some α to be determined, for example,

$$p(x,t) = \frac{1}{\alpha}\int_{-\infty}^{\infty} \pi(x-y,t)\rho_0(y)\mathrm{d}y - \int_{-\infty}^{\infty}\int_0^t \pi(x-y,t-s)\frac{1}{\alpha}\left(|w(y,s)| + \bar{\rho}(y,s)\right)\mathrm{d}s\mathrm{d}y.$$

Then, since $\frac{w}{\bar{\rho}} \in L^\infty$, we can choose α sufficiently large so that

$$\frac{1}{\alpha}\bar{\rho}(x,t) \leq \frac{1}{2}p_{xx}(x,t) - p_t(x,t) \leq \bar{\rho}(x,t).$$

In particular, since $\frac{1}{2}\bar{\rho}_{xx}(x,t) - \bar{\rho}_t(x,t) = \lambda\bar{\rho}(x,t)$, this means that

$$\frac{1}{\lambda}\bar{\rho}(x,t) \leq p(x,t) \leq \frac{1}{\alpha\lambda}\bar{\rho}(x,t),$$

and so $p(x,t) \geq 0$ for all (x,t), and $\frac{p}{\bar{\rho}} \in L^\infty$. Then, since $\frac{(\frac{1}{2}p_{xx} - p_t)(x,t)}{\bar{\rho}(x,t)} = \frac{1}{\alpha}\left(\frac{|w(x,t)|}{\bar{\rho}(x,t)} + 1\right)$ $\in L^\infty$, we have $p \in \mathsf{X}$, as required.

Furthermore,

$$y(x,t) = \frac{1}{\alpha}\left(w(x,t) + |w(x,t)| + \bar{\rho}(x,t)\right) \geq 0,$$

and since $\frac{z}{L_\mu^{\cdot}}, \frac{\int_0^\infty \rho(\cdot,t)\mathrm{d}t}{L_\mu^{\cdot}} \in L^\infty$ we can choose α large so that

$$\begin{aligned}\bar{y}(x) &= \frac{1}{\alpha}z(x) - \int_0^\infty p(x,t)\mathrm{d}t \\ &\geq \frac{1}{\alpha}z(x) - \frac{1}{\alpha\lambda}\int_0^\infty \bar{\rho}(x,t)\mathrm{d}t \\ &\geq -L_\mu^x.\end{aligned}$$

Finally, $y_0(x) = \frac{1}{\alpha}(w_0(x) - \rho_0(x)) \geq -\frac{1}{\alpha}\left(\|\frac{w_0}{\rho_0}\|_{L^\infty} + 1\right)\rho_0(x) \geq -\rho_0(x)$ for large α. □

4.5 Duality in Unweighted Space

We have shown that $\mathsf{p}(\lambda) = \mathsf{d}(\lambda)$, with dual value attained, however our goal is to show that $\tilde{\mathsf{p}} = \tilde{\mathsf{d}}$.

4.5.1 Primal Equality

We have some obvious inclusions from the definitions of our primal spaces, namely

$$\begin{aligned}\mathcal{P}(\lambda) &\subseteq \mathcal{P} \\ \tilde{\mathcal{P}}(\lambda) &\subseteq \tilde{\mathcal{P}}.\end{aligned}$$

Note that $\mathsf{X} \not\subseteq \widetilde{\mathsf{X}}$ since although any $p \in \mathcal{P}$ defines a stopping measure with density q with $\|q\|_\infty < \infty$, we do not necessarily have that $\|q\|_{TV} < \infty$. However, we do have $\mathcal{P} \subseteq \tilde{\mathcal{P}}$ since any $p \in \mathcal{P}$ gives a stopping measure with density q where $\|q\|_{TV} \leq 1$. Therefore,

$$\begin{aligned}\mathcal{P} &\subseteq \tilde{\mathcal{P}} \\ \mathcal{P}(\lambda) &\subseteq \tilde{\mathcal{P}}(\lambda) \\ \mathcal{P}(\lambda) &\subseteq \tilde{\mathcal{P}}.\end{aligned}$$

These inclusions imply that

$$\begin{aligned}\mathsf{p}(\lambda) \leq \tilde{\mathsf{p}}(\lambda) &\leq \tilde{\mathsf{p}}, \\ \mathsf{p}(\lambda) \leq \quad \mathsf{p} \quad &\leq \tilde{\mathsf{p}}.\end{aligned}$$

To move between the weighted and unweighted spaces we can use a 'cut-off' argument, as in Section 3.5.2, but we also need a way to move between $\mathcal{P}$ and the more difficult space $\tilde{\mathcal{P}}$. In $\tilde{\mathsf{X}}$ we allow stopping distributions such as delta functions, corresponding to some points or regions where we stop all paths. At these points we could, instead of stopping the paths, introduce some external randomisation and allow the paths to run for some uniform time longer on $[0, \varepsilon]$ for small ε. This 'smooths out' the stopping distribution so that we have a density in W, and we can therefore relate p and $\tilde{\mathsf{p}}$, or $\mathsf{p}(\lambda)$ and $\tilde{\mathsf{p}}(\lambda)$.

Lemma 4.17. *The primal values are related by*

$$\tilde{\mathsf{p}} = \tilde{\mathsf{p}}(\lambda) = \mathsf{p}(\lambda) = \mathsf{p}.$$

Proof. We use the techniques of Lemma 4.13. Let $\varepsilon > 0$ be small. Take any (randomised) stopping rule, with stopping time τ_p, corresponding to a feasible $p \in \tilde{\mathcal{P}}(\lambda)$, and for each path simulate an independent uniform random variable, U on $(0,1)$. If, under τ_p, a path is stopped at (x,t), for $x \in (x_*, x^*)$, allow the path to continue running until it exits the box

$$\left\{(y,s) \in \overline{\mathsf{S}} : \ |x-y| < U, \ s-t < U \frac{\varepsilon}{1-\varepsilon} \inf_{z \in \mathcal{I}_x} L^z_\mu \right\},$$

where $\mathcal{I}_x := (x - \frac{1}{2}, x + \frac{1}{2}) \cap (\frac{1}{2}(x^* + x), \frac{1}{2}(x_* + x))$. Note that this only smooths out paths stopped in $(x_*, x^*) \times [0, \infty)$, but no element of $\tilde{\mathcal{P}}(\lambda)$ allows stopping at x_* or x^*. This smooths out any nasty stopping distributions that embed μ and gives $\bar{p}$ such that $\int_0^\infty \bar{p}(x,t)\mathrm{d}t \le L^x_\varepsilon$ for all $x \in [x_*, x^*]$, where L_ε is such that

$$(1-\varepsilon)L^x_{\mu^\varepsilon} \le L^x_\mu. \tag{4.2}$$

Consider the primal problem $\mathcal{P}^\varepsilon$ which is identical to $\mathcal{P}$ except that we use the measure μ^ε in place of μ. Similarly we can define $\mathcal{P}^\varepsilon(\lambda)$ and $\mathsf{p}^\varepsilon, \mathsf{p}^\varepsilon(\lambda)$. The arguments of Lemma 4.13 show

$$\tilde{\mathsf{p}} \le \mathsf{p}^\varepsilon$$
$$\mathsf{p} \le \mathsf{p}^\varepsilon$$
$$\tilde{\mathsf{p}}(\lambda) \le \mathsf{p}^\varepsilon(\lambda).$$

Also, if we take any $\bar{p} \in \mathcal{P}^\varepsilon$, then by (4.2), $p := (1-\varepsilon)\bar{p} \in \mathcal{P}$, and this again holds for the weighted spaces. In this case we see that $\Phi(p) = (1-\varepsilon)\Phi(\bar{p})$, and taking supremums gives

$$(1-\varepsilon)\mathsf{p}^{\varepsilon} \leq \mathsf{p},$$
$$(1-\varepsilon)\mathsf{p}^{\varepsilon}(\lambda) \leq \mathsf{p}(\lambda).$$

We have already seen that $\mathsf{p}(\lambda) \leq \mathsf{p}$, $\tilde{\mathsf{p}}(\lambda) \leq \tilde{\mathsf{p}}$, and $\mathsf{p}(\lambda) \leq \tilde{\mathsf{p}}$, and we can in fact show the opposite inequalities also. Suppose for example that we want to show that $\tilde{\mathsf{p}} \leq \tilde{\mathsf{p}}(\lambda)$. Fix $p \in \tilde{\mathcal{P}}$, and the corresponding measure q. We don't, in general, know that p, q have the necessary exponential decay for p to be an element of $\tilde{\mathcal{P}}(\lambda)$, but we can instead consider 'cutting-off' p at some finite time by defining

$$p_n(x,t) = \begin{cases} p(x,t), & t \leq n \\ p(x,t)\mathrm{e}^{-(\lambda+1)(t-n)}, & t > n \end{cases} \tag{4.3}$$

for any $n \in \mathbb{N}$. Then we have $(p_n)_n \in \tilde{\mathcal{P}}(\lambda)$ and $\Phi(p_n) \to \Phi(p)$ as $n \to \infty$. Therefore, $\tilde{\mathsf{p}}(\lambda) \geq \tilde{\mathsf{p}}$.

Combining these inequalities, we find

$$\mathsf{p}(\lambda) \leq \tilde{\mathsf{p}} \leq \tilde{\mathsf{p}}(\lambda) \leq \mathsf{p}^{\varepsilon}(\lambda) \leq \frac{1}{1-\varepsilon}\mathsf{p}(\lambda).$$

Since these hold for any $\varepsilon > 0$, the result follows. $\square$

4.5.2 Dual Equality

Recall that our dual problem (when the primal problem is taken over $\mathcal{P}(\lambda)$), is

$$\inf_{\eta,\nu} \Psi(\eta,\nu) := \int L^x_\mu \nu(\mathrm{d}x) + \int \rho_0(x)F(x,0)\mathrm{d}x + \int \rho_0(x)\eta_0(\mathrm{d}x)$$

over measures η on S, and ν on (x_*, x^*), subject to

- $(\eta,\nu) \in \mathsf{Y}(\lambda)^*$
- $\eta(D) \geq_{w(\overline{\mathsf{S}})} 0, \nu(E)_{w([x_*,x^*])} \geq 0$ (D1)
- $\frac{1}{2}\eta_{xx} + \eta_t - \tilde{\nu} \leq_{w(\mathsf{S})} -\left(\frac{1}{2}F_{xx} + F_t\right),$ (D2)

where $\tilde{\nu}(\mathrm{d}x,\mathrm{d}t) = \nu(\mathrm{d}x)\mathrm{d}t$ and $\eta_0(\mathrm{d}x) = \eta(\mathrm{d}x \times \{0\})$.

For this problem we have an optimal pair of finitely additive measures $(\eta^*, \nu^*) \in \mathsf{Y}(\lambda)^*$ such that $\Psi(\eta^*, \nu^*) = \mathsf{d}(\lambda) = \mathsf{p}(\lambda) = \tilde{\mathsf{p}}$. We now hope to show that $\mathsf{d}(\lambda) = \tilde{\mathsf{d}}$.

First we note that we have weak duality in our remaining three optimality problems, and therefore

$$\begin{aligned} \mathsf{p} &\leq \mathsf{d}, \\ \tilde{\mathsf{p}} &\leq \tilde{\mathsf{d}}, \\ \tilde{\mathsf{p}}(\lambda) &\leq \tilde{\mathsf{d}}(\lambda). \end{aligned}$$

We also know have the primal inclusions $\mathcal{P}(\lambda) \subseteq \mathcal{P}$ and $\tilde{\mathcal{P}}(\lambda) \subseteq \tilde{\mathcal{P}}$. Note that the cut-off arguments of (4.3) show that these are dense subsets, and therefore by Theorem 4.10 we have

$$\begin{aligned} \mathcal{D} &\subseteq \mathcal{D}(\lambda), \\ \tilde{\mathcal{D}} &\subseteq \tilde{\mathcal{D}}(\lambda). \end{aligned}$$

These then imply the following:

$$\begin{aligned} \mathsf{d}(\lambda) &\leq \mathsf{d}, \\ \tilde{\mathsf{d}}(\lambda) &\leq \tilde{\mathsf{d}}, \end{aligned}$$

and so we have

$$\begin{aligned} \tilde{\mathsf{p}} = \tilde{\mathsf{p}}(\lambda) = \mathsf{p}(\lambda) = \mathsf{d}(\lambda) \leq \tilde{\mathsf{d}}(\lambda) \leq \tilde{\mathsf{d}} \\ \tilde{\mathsf{p}} = \tilde{\mathsf{p}}(\lambda) = \mathsf{p}(\lambda) = \mathsf{d}(\lambda) \leq \mathsf{p} \leq \mathsf{d} \end{aligned}$$

Next we relate $\tilde{\mathsf{d}}$ and d by approximating both problems using sets of continuous functions. In particular we consider dual triples $(f, f_0, g) \in C_b(\mathsf{S}) \times \mathsf{V}(\rho_0) \times \mathsf{V}(L_\mu)$, where

$$\begin{aligned} \mathsf{V}(L_\mu) &:= \left\{ f \middle| \, f \cdot L_\mu \in C_b((x_*, x^*)) \right\} \\ \mathsf{V}(\rho_0) &:= \left\{ g \middle| \, g \cdot \rho_0 \in C_b(\mathsf{T}) \right\}. \end{aligned}$$

As with $\tilde{\mathcal{D}}$ we can consider $f_0(x)$ as $f(x, 0)$. Define

$$\tilde{\mathcal{C}} := \{(\eta, \eta_0, \nu) \in C_b(\mathsf{S}) \times \mathsf{V}(\rho_0) \times \mathsf{V}(L_\mu) : \ (\eta, \eta_0, \nu) \text{ satisfy (D1), (D2)}\},$$

then we have the following.

Lemma 4.18. *The dual problem $\tilde{\mathcal{D}}$ can be approximated by optimising over continuous functions, and*

$$\tilde{\mathsf{d}} = \inf_{(f,f_0,g)\in\tilde{\mathcal{C}}} \Psi(f,g) \leq \mathsf{d}.$$

Proof. By Theorem 4.7, we have that $C_0(\mathsf{S})$ is weakly-* dense in $C_0(\mathsf{S})^{**} = \widetilde{\mathsf{W}}^*$. Then, for any function $\eta \in \widetilde{\mathsf{W}}^*$ and $\varepsilon > 0$, there exists $f^\varepsilon \in C_0(\mathsf{S})$ such that for any $\sigma \in \widetilde{\mathsf{W}}$,

$$|\langle f^\varepsilon, \sigma\rangle - \langle \eta, \sigma\rangle| = \left| \int f^\varepsilon(x,t)\sigma(\mathrm{d}x,\mathrm{d}t) - \eta(\sigma)\right| < \varepsilon.$$

Any test function $\varphi \in C_0^\infty(\mathsf{S})$ defines a measure $\sigma_\varphi \in \tilde{W}$, and furthermore so does $\frac{1}{2}\varphi_{xx} - \varphi_t$. Then for such a function φ we have

$$\left|\langle \frac{1}{2}f^\varepsilon_{xx} + f^\varepsilon_t - \frac{1}{2}\eta_{xx} - \eta_t, \varphi\rangle\right| := \left|\langle f^\varepsilon - \eta, \sigma_{\frac{1}{2}\varphi_{xx}-\varphi_t}\rangle\right| < \varepsilon,$$

and so we have convergence of the derivatives in the distributional sense also. Similarly, $\{f \big| \, f\cdot L_\mu \in L^1((x_*,x^*))\}$ is weakly-* dense in Z^*, and $\{g \big| \, g\cdot\rho_0 \in L^1(\mathsf{T})\}$ is weakly-* dense in ${\mathsf{W}_0}^*$. Also, $C_b((x_*,x^*))$ is dense in $L^1((x_*,x^*))$ and it is simple to check that $\mathsf{V}(L_\mu) := \{f \big| \, f\cdot L_\mu \in C_b((x_*,x^*))\}$ and $\mathsf{V}(\rho_0) := \{g\big| \, g\cdot\rho_0 \in C_b(\mathsf{T})\}$ are weakly-* dense in Z^* and ${\mathsf{W}_0}^*$ respectively. Note also that $C_0(\mathsf{S}) \subseteq C_b(\mathsf{S})$.

For $\delta \geq 0$, define $\tilde{\mathcal{D}}^\delta$ to be the set of triplets $(f, f_0, g) \in C_b(\mathsf{S}) \times \mathsf{V}(\rho_0) \times \mathsf{V}(L_\mu)$ such that

- $f \geq -\delta$, $f_0 \geq -\delta$, $g \geq -\delta$
- $\frac{1}{2}f_{xx} + f_t - g \leq_{w(\mathsf{S})} -\left(\frac{1}{2}F_{xx} + F_t\right) + \delta.$

Then for any $(\eta, \eta_0, \nu) \in \tilde{\mathcal{D}}$, and $\delta > 0$, there exists $(\eta^\delta, \eta_0^\delta, \nu^\delta) \in \tilde{\mathcal{D}}^\delta$ such that $\left|\Psi(\eta^\delta,\eta_0^\delta,\nu^\delta) - \Psi(\eta,\eta_0,\nu)\right| < \delta$. If $\mathsf{d}^\delta := \inf_{(f,f_0,g)\in\tilde{\mathcal{D}}^\delta} \Psi(f, f_0, g)$, then the above implies that $\lim_{\delta\to 0}\mathsf{d}^\delta \leq \tilde{\mathsf{d}}$. Note also that d^δ is decreasing in δ, and that $\tilde{\mathcal{D}}^0 \subseteq \tilde{\mathcal{D}}$, since $C_b(\mathsf{S}) \subseteq rca(\mathsf{S})^*$ and $C_b((x_*,x^*)) \subseteq L^\infty((x_*,x^*))^*$, so

$$\lim_{\delta\to 0}\mathsf{d}^\delta \leq \tilde{\mathsf{d}} \leq \mathsf{d}^0.$$

Furthermore, for any $(f, f_0, g) \in \tilde{\mathcal{D}}^\delta$ we can define $\bar{f} := f+\delta$, $\bar{f}_0 := f_0+\delta$, and $\bar{g} := g+\delta$

so that

$$\bar{f} \geq 0,\ \bar{f}_0 \geq 0,\ \bar{g} \geq 0,$$

and

$$\frac{1}{2}\bar{f}_{xx} + \bar{f}_t - \bar{g} = \frac{1}{2}f_{xx} + f_t - g - \delta \leq_{w(\mathsf{S})} -\left(\frac{1}{2}F_{xx} + F_t\right),$$

and therefore $(\bar{f}, \bar{f}_0, \bar{g}) \in \tilde{\mathcal{D}}^0$. Also note that

$$\left|\Psi(f, f_0, g) - \Psi(\bar{f}, \bar{f}_0, \bar{g})\right| = \delta \int_{x_*}^{x^*} \left(L_\mu^x + \rho_0(x)\right) \mathrm{d}x \leq C\delta$$

for some constant $0 < C < \infty$ independent of δ. Then $\mathsf{d}^0 \leq \lim_{\delta \to 0} \mathsf{d}^\delta$, and so $\tilde{\mathsf{d}} = \mathsf{d}^0$.

Consider now $(\eta, \eta_0, \nu) \in \mathcal{D}$, so $\eta \in \mathsf{W}^* = L^\infty(\mathsf{S})^*$. From Theorem 4.7 we know that $L^1(\mathsf{S})$ is weakly-* dense in W^*, and then by the same arguments as above we deduce that $C_b(\mathsf{S}) \cap \mathsf{W}^*$ is also weakly-* dense in W^*, and $\mathsf{d}^0 = \lim_{\delta \to 0} \mathsf{d}^\delta \leq \mathsf{d}$. Note that $C_b(\mathsf{S})$ is not a subset of W^*, so we do not necessarily get equality. However, we have shown that

$$\tilde{\mathsf{d}} \leq \mathsf{d}.$$

It is easy to check that the above arguments also apply to the weighted spaces, so

$$\tilde{\mathsf{d}}(\lambda) \leq \mathsf{d}(\lambda).$$

□

We now apply cut-off arguments to relate the weighted and unweighted problems. For example, given a dual-feasible pair $(\eta, \nu) \in \tilde{\mathcal{D}}(\lambda)$ we wish to find a sequence $(\eta^n, \nu^n) \in \tilde{\mathcal{D}}$ such that $\Psi(\eta^n, \nu^n) \to \Psi(\eta, \nu)$ as $n \to \infty$, so that $\tilde{\mathsf{d}} \leq \tilde{\mathsf{d}}(\lambda)$. Since we are working on a bounded domain, we have that $F(x,t) = 0$ for all $x \in [x_*, x^*]$ when $t \geq K(x^*)$ and, as we saw in Chapter 3, we have some freedom in our stopping region after $K(x^*)$. In the discrete setup we used this feature of the payoff to cut-off the dual variables past $K(x^*)$, and we do the same here.

Lemma 4.19. *Any $(\eta, \nu) \in \tilde{\mathcal{D}}(\lambda)$ have unweighted versions $(\eta^n, \nu^n) \in \tilde{\mathcal{D}}$ such that $\Psi(\eta^n, \nu^n) = \Psi(\eta, \nu)$, so*

$$\tilde{\mathsf{d}} \leq \tilde{\mathsf{d}}(\lambda).$$

Proof. Take any $(\eta, \nu) \in \tilde{\mathcal{D}}(\lambda)$ and for $n \in \mathbb{N}$ define $\nu^n \equiv \nu \in \mathsf{W_0}^*$, and let $\rho^n \in C_0^\infty([0, \infty))$ be any decreasing approximation of $\mathbf{1}\{t \leq K(x^*) + 1\}$ such that $\rho^n(t) = 1$

for $t < K(x^*) + 1 - \frac{1}{n}$ and $\rho^n(t) = 0$ for $t > K(x^*) + 1$. Define $\eta^n \in \widetilde{\mathsf{W}}^*$ to be the functional such that $\eta^n(q) = \eta(\rho^n q)$ for any $q \in rca(\overline{\mathsf{S}})$. Note that $\rho^n q \in \widetilde{\mathsf{W}}(\lambda)$ for any $q \in rca(\overline{\mathsf{S}})$ and so this does define an element of $\widetilde{\mathsf{W}}^*$ for each n. Also, for any $\varphi \in C_0^\infty(\mathsf{S})$, we have that $\rho^n\varphi \in C_0^\infty(\mathsf{S})$ is also a test function.

We have seen in Lemma 4.18 that there exist functions $f^\varepsilon \in C_0(\mathsf{S})$ converging in the weak-* sense to η, and that their derivatives converge weakly as distributions. For any $q \in \widetilde{\mathsf{W}}$,

$$|\langle f^\varepsilon \rho^n, q\rangle - \langle \eta^n, q\rangle| = |\langle f^\varepsilon, \rho^n q\rangle - \langle \eta, \rho^n q\rangle| < \varepsilon,$$

and so $(f^\varepsilon \rho^n)_\varepsilon \in C_0(\mathsf{S})$ converges in the weak-* sense to η^n for any n. The derivatives must also then converge, as in Lemma 4.18.

For any $\varepsilon > 0$, n, and $\varphi \in C_0^\infty(\mathsf{S})$ with $\varphi \geq 0$,

$$\begin{aligned}
\left\langle \frac{1}{2}(f^{\frac{\varepsilon}{3}}\rho^n)_{xx} + (f^{\frac{\varepsilon}{3}}\rho)^n_t, \varphi \right\rangle &= \left\langle \frac{1}{2} f^{\frac{\varepsilon}{3}}_{xx} + f^{\frac{\varepsilon}{3}}_t, \rho^n\varphi \right\rangle + \left\langle f^{\frac{\varepsilon}{3}}\rho^n_t, \varphi \right\rangle \\
&\leq \left\langle \frac{1}{2} f^{\frac{\varepsilon}{3}}_{xx} + f^{\frac{\varepsilon}{3}}_t, \rho^n\varphi \right\rangle + \langle \eta, \rho^n_t\varphi\rangle + \frac{\varepsilon}{3} \\
&\leq \left\langle \frac{1}{2} f^{\frac{\varepsilon}{3}}_{xx} + f^{\frac{\varepsilon}{3}}_t, \rho^n\varphi \right\rangle + \frac{\varepsilon}{3}
\end{aligned}$$

since $\eta \geq_{w(\mathsf{S})} 0$ and $\rho^n_t\varphi \leq 0$. Therefore

$$\begin{aligned}
\left\langle \frac{1}{2}\eta^n_{xx} + \eta^n_t, \varphi \right\rangle &\leq \left\langle \frac{1}{2} f^{\frac{\varepsilon}{3}}_{xx} + f^{\frac{\varepsilon}{3}}_t, \rho^n\varphi \right\rangle + \frac{2\varepsilon}{3} \\
&\leq \left\langle \frac{1}{2}\eta_{xx} + \eta_t, \rho^n\varphi \right\rangle + \varepsilon \\
&\leq \left\langle \tilde{\nu} - \left(\frac{1}{2}F_{xx} + F_t\right), \rho^n\varphi \right\rangle + \varepsilon \\
&= \langle \tilde{\nu}^n, \varphi\rangle - \langle \tilde{\nu}, \varphi(1-\rho^n)\rangle - \left\langle \frac{1}{2}F_{xx} + F_t, \rho^n\varphi \right\rangle + \varepsilon \\
&\leq \left\langle \tilde{\nu}^n - \left(\frac{1}{2}F_{xx} + F_t\right), \varphi \right\rangle + \varepsilon,
\end{aligned}$$

since $\tilde{\nu} \geq_{w(\mathsf{S})} 0$ and $\frac{1}{2}F_{xx} + F_t \equiv 0$ on $\{t > K(x^*)\}$. Since this holds for any $\varepsilon > 0$, we have $(\eta^n, \nu^n) \in \tilde{\mathcal{D}}$.

Note finally that $\Psi(\eta^n, \nu^n) = \Psi(\eta, \nu)$, so $\tilde{\mathsf{d}} \leq \tilde{\mathsf{d}}(\lambda)$. $\square$

Finally then, we have proved the following.

Theorem 4.20. *When* $F(x,t) = (\mathrm{e}^{\beta x - \frac{1}{2}\beta^2 t} - k)_+$, *we have*

$$\tilde{\mathsf{p}} = \tilde{\mathsf{d}},$$

and $\tilde{\mathsf{p}}$ *is attained by some* p^* *corresponding to a stopping time for a Brownian motion which embeds* μ.

4.5.3 Cut-off Arguments for Other Payoffs

The aim of the cut-off argument is to let η^n be a smooth approximation of $\eta(x,t)\mathbf{1}\{t \leq n, |x| \leq n\}$, for example $\eta^n = \eta \cdot \varphi^n$ or $\eta^n = \eta * \varphi^n$ for some $\varphi^n \in C_0^\infty(\mathsf{S})$ with $\varphi \equiv 1$ on $\{t \leq n, |x| \leq n\}$ and $\varphi \equiv 0$ on $\{t > n+\delta, |x| > n+\delta\}$ for small $\delta > 0$. These choices would ensure that η^n is compactly supported, and therefore no longer has exponential growth, so $\eta^n \in \tilde{\mathsf{W}}$. However, we seem unable to find a suitable φ^n and ν^n to ensure that the derivative condition $\frac{1}{2}\eta^n_{xx} + \eta^n_t - \nu^n \leq_{w(\mathsf{S})} -(\frac{1}{2}F_{xx} + F_t)$ is satisfied for general F.

If $F \in C^1([0,\infty))$ is a function of time only, and there exists n such that $F'(t) \leq 0$ for $t > n$, then we can use $\eta^n(x,t) = \eta(x,t)\mathbf{1}\{t \leq n\}$ as above, and this will be the case in some examples of the Root and Rost embeddings. In the case of the cave embedding, we have that for $t > t_0$, $F(t)$ is concave, increasing, but $F(t) \to 0$ as $t \to \infty$, so for all $n \in \mathbb{N}$ there is some T_n such that $-\frac{1}{n} < F'(t) \leq 0$ for $t \geq T_n$. Then we can take $\eta^n(x,t) = \eta(x,t)\mathbf{1}\{t \leq n\}$ and $\nu^n = \nu + \frac{1}{n}$, to find a sequence $(\eta^n, \nu^n) \in \tilde{\mathcal{D}}$ such that $\Psi(\eta^n, \nu^n) \to \Psi(\eta, \nu)$ as $n \to \infty$.

4.6 Conclusions

We have reproved duality in the case of the K-cave embedding, but have highlighted that we should not in general expect dual attainment. For the K-cave problem we have found dual optimisers when optimising over the sets $\mathcal{P}(\lambda)$ and $\mathcal{D}(\lambda)$, but to deduce that these are also elements of $\tilde{\mathcal{D}}$ requires some extra properties of the problem.

In the barrier-type solutions of (OptSEP) we consider in this thesis, we expect an optimal stopping time τ with corresponding $p_\tau \in \tilde{\mathcal{P}} \setminus \mathcal{P}(\lambda)$ and in Chapter 3 we find dual optimisers G, H of (OptSEP) which are defined in terms of the optimal stopping time τ. Since there may not be a primal optimiser in $\mathcal{P}(\lambda)$, any optimal $(\eta, \nu) \in \mathcal{D}(\lambda)$ will not be defined in terms of an optimal stopping time, and so it is unlikely that

$(\eta, \nu) \in \tilde{\mathcal{D}} \subseteq \mathcal{D}(\lambda)$.

In other words, we have proven in Lemma 4.14 and Theorem 4.16 that we have $p \in \tilde{\mathcal{P}}$ and $(\eta, \nu) \in \mathcal{D}(\lambda)$ such that

$$\tilde{\mathsf{p}} = \Phi(p) = \Psi(\eta, \nu) = \mathsf{d}(\lambda).$$

However, $\eta \in \mathsf{W}(\lambda)^*$ is not in the dual space of $\frac{1}{2}p_{xx} - p_t \in \tilde{W}$, and so unlike in the discrete arguments of Lemma 3.10, we *cannot* use the complementary slackness conditions to find the form of η in terms of the optimal p.

Moving between $\tilde{\mathcal{P}}$ and $\mathcal{P}(\lambda)$ requires a weighting, to ensure the mass decays exponentially, and also some smoothing of the measures to ensure the stopping measures have densities. We propose that the exponential decay property of any optimiser is necessary for strong duality in full generality, however we believe that the smoothing condition is merely a technical assumption that we have made in order to prove strong duality in some space. This can be summarised as the following conjecture.

Conjecture. *We expect that for any bounded payoff F, a strong duality result such as Theorem* 4.16 *holds for $\tilde{\mathcal{P}}(\lambda)$ and $\tilde{\mathcal{D}}(\lambda)$ so that*

$$\tilde{\mathsf{p}} = \tilde{\mathsf{d}},$$

and $\tilde{\mathsf{d}}$ is attained by some $(\eta, \nu) \in \tilde{\mathcal{D}}(\lambda)$.

If the above conjecture is correct, then in the K-cave case we have the strong duality result we require, and we would expect to find that η corresponds to G^* from Section 3.4. For payoffs other than the option on the leveraged exchanged fund, we will still have optimal $(\eta, \nu) \in \tilde{\mathcal{D}}(\lambda)$, but the cut-off arguments of Lemma 4.19 will not necessarily be valid. It appears that we are able to move from $\tilde{\mathcal{D}}(\lambda)$ to $\tilde{\mathcal{D}}$ in the cases where we expect the optimal p to be in $\tilde{\mathcal{P}}(\lambda)$. This is clear in the cases of the Root, cave, and K-cave embeddings since the existence of a Root barrier ensures that for any embedding stopping time τ, $\mathbb{P}(\tau \geq t)$ decays exponentially faster than $\mathbb{P}(H_{x_*}(W) \wedge H_{x^*}(W) \geq t)$.

In the case of the Rost embedding (in particular where we choose μ to have atoms at x_* and x^*), we will not expect this decay. This means that any optimal p is likely to be in $\tilde{\mathcal{P}} \setminus \tilde{\mathcal{P}}(\lambda)$, and we therefore expect dual optimisers in $\tilde{\mathcal{D}}(\lambda) \setminus \tilde{\mathcal{D}}$, so the cut-off will not work. The dual optimisers of (OptSEP) for the Rost problem are found in Cox and Wang [2013b], and the results there show the same restrictions. The authors are able to give dual superhedging functions when they restrict the problem by introducing some finite time horizon T, but to attain global optimisers they require extra assumptions on the payoff function F, namely that there exist constants $C > 0$ and $\alpha > 0$ such that for large t, $C \geq F'(t) \geq C - t^{-\alpha}$. This condition is necessary to prove the existence of the candidate dual optimiser $G(x,t) + H(x) = \int_t^\infty (M(x,s) - F'(s))\,\mathrm{d}s$, where $M(x,s) := \mathbb{E}^{x,s}[F'(\tau)]$, as otherwise the integral does not converge. However, similarly to the reasoning above, this condition actually ensures that any optimal stopping will have the exponential decay property we require. In the notation of this chapter, the above G and H from Cox and Wang [2013b] give optimal $(\eta,\nu) \in \tilde{\mathcal{D}}(\lambda)$ and the condition on the payoff function implies that any optimal p lies in $\tilde{\mathcal{P}}(\lambda)$, and therefore that the dual optimal $(\eta,\nu) \in \tilde{\mathcal{D}}(\lambda)$ are true duals to p. The condition also ensures that $(\eta,\nu) \in \tilde{\mathcal{D}}$, and so it appears that the condition for moving between the weighted and unweighted spaces is equivalent in both the primal and dual problems.

In summary, we believe that the arguments of this chapter, and of Chapter 2, show that the existence of dual optimisers is dependent on this exponential decay of the non-embedded mass, as this ensures that any candidate optimisers G, H are indeed defined in terms of an optimal stopping time τ. For a well-behaved payoff function there is always an optimal $p \in \tilde{\mathcal{P}}$, and we expect that there are always optimal $(\eta,\nu) \in \tilde{D}(\lambda)$. If the payoff is such that $p \in \tilde{\mathcal{P}}(\lambda)$, then we can make sense of $\left\langle \eta, \frac{1}{2}p_{xx} - p_t \right\rangle$ and can find our superhedging strategy. The conditions under which $p \in \tilde{\mathcal{P}}(\lambda)$ should be equivalent to those which ensure that $(\eta,\nu) \in \tilde{D}$, and so we can move between the weighted and unweighted problems freely.

Chapter 5

Optimal Skorokhod Embeddings as Stochastic Optimal Control Problems

In this chapter we give another reformulation of the optimal Skorokhod embedding problem, now as a stochastic optimal control problem. We first show the equivalence of a modified version of (OptSEP) and a forward-backward stochastic differential equation, the solutions to which can be thought of as controlled processes. The optimisation over these is then a stochastic optimal control problem, and the stochastic maximum principle applies.

In the example of the Root embedding the optimal control corresponds to a Root barrier and the embedding constraint is enforced through a secondary optimisation. This secondary problem penalises the accumulation of local time, and the optimisation becomes a balance between improving the payoff but minimising the accrued local time. We see that this has comparisons to the dual problem of the Root embedding, and we propose that in the case of the cave and K-cave embeddings the equilibrium point of this balancing act gives the condition (Γ) from Chapter 3.

5.1 Introduction

We have seen that the problem of finding model-independent bounds on the price of financial derivatives is an optimisation over a set of feasible probability measures, or a family of stopping times, and can therefore be viewed as a stochastic control problem. This relation is exploited in Galichon et al. [2014] to prove the duality of the superhedging problem, and in this paper the control problem is linked naturally to the dynamic programming principle and a free boundary problem. The link between the Skorokhod embedding problem and free boundary problems is also explored in Cox and Wang [2013a,b], Gassiat et al. [2015], and Cox et al. [2018]. In particular, in Gassiat et al. [2015] the embedding problem, and associated free boundary problem, is linked to a particular FBSDE, and we prove such a relation in Theorem 5.1.

Our aim is to rewrite (OptSEP) as an optimisation problem constrained by BSDEs, which can then be solved using ideas from control theory. With this in mind, we consider the modified version of the above problem where we have a finite time horizon T and require only an inequality in the embedding condition. We again work on a filtered probability space $(\Omega, \mathbb{P}, \mathcal{F}, (\mathcal{F}_t)_t)$ that is rich enough to support a Brownian motion W and a uniformly distributed random variable U independent of W. Equivalently, we can think of our probability measure as $\mathbb{P} = \mathbb{P}_W \times \mathbb{P}_U$, where $\mathbb{P}_W$ is the Wiener measure, and $\mathbb{P}_U$ is the Lebesgue measure on $[0,1]$. We will denote the expectations with respect to these measures by $\mathbb{E}^W$ and $\mathbb{E}^U$ respectively, and as usual $\mathbb{E}$ will represent expectation with respect to $\mathbb{P}$. For a centered probability distribution μ with finite second moment, we wish to solve

$$\sup_{\tau} \mathbb{E}\left[F(W_\tau, \tau)\right] \tag{OptSEP'}$$

over randomised stopping times τ such that

- $\tau \leq T$
- $\mathbb{E}\left[L_\tau^x\right] \leq w_\mu(x) := U_{\delta_0}(x) - U_\mu(x) \quad \forall x$
- $(W_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable.

The bound on the stopping time is required for technical reasons, however for large T the optimal solution will change very little. In particular for any $\varepsilon > 0$, by Markov's inequality, there is some T such that $\mathbb{P}(\tau > T) < \varepsilon$ for any stopping time τ such that $W_\tau \sim \mu$.

The inequality constraint allows us to stop before we have the correct embedding, i.e. we can embed a distribution smaller than or equal to μ in convex order. Note that if $F(W_t, t)$ is a submartingale, then it will always be optimal to run our Brownian motion for as long as possible before stopping, so (provided an optimiser exists) there will always be an optimiser with equality in the local time condition. If we instead wished to consider a function F such that $F(W_t, t)$ is a supermartingale, then we could consider the opposite inequality, and for general cases we may wish to bound the expected local time between two values either side of the desired amount. Also note that, by Proposition 1.9, we only have to enforce this condition on the support of μ.

We can now relate this adjusted problem to a BSDE problem. Let $\mathcal{F}_t^W$ be the natural filtration of W, and for a randomised stopping time τ, define a process R_t by

$$R_t := \mathbb{E}\left[\mathbf{1}\{\tau \leq t\} \middle| \mathcal{F}_t^W\right].$$

Then R_t is an increasing, $\mathcal{F}_t^W$-measurable process in $[0,1]$ with $R_0 = 0$. Note that $\mathbf{1}\{\tau \leq t\}$ is not necessarily $\mathcal{F}_t^W$-measurable since we could have some extra randomisation in the randomised stopping time.

Suppose now that μ has bounded support with $x_* = \sup\{x : \mu((-\infty, x)) = 0\}$ and $x^* := \inf\{x : \mu((x, \infty)) = 0\}$ and let $\mathbb{D} := \{z_1, z_2, \ldots\}$ be a dense, countable subset of $[x_*, x^*]$ which contains x_* and x^*. Then we have the following result.

Theorem 5.1. *If* $\mathbb{E}\left[F(W_T, T)^2\right] < \infty$ *and* $\mathbb{E}\left[\int_0^T (\mathcal{L}F(W_s, s))^2 \mathrm{d}s\right] < \infty$*, then the following are equivalent:*

1. *there exists a randomised solution,* τ*, to* (OptSEP') *with optimal value* x_0*,*

2. there exists a solution $(X, Y(z), Z, \xi(z), R)$ *to*

- $X_0 = x_0,$ (A1)
- $X_t = -\int_t^T R_s \mathcal{L}F(W_s, s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s + F(W_T, T), \quad \forall 0 \le t \le T$ (A2)
- $Y_t(z) = w_\mu(z) - \int_0^t (1 - R_s)\mathrm{d}L_s^z + \int_0^t \xi_s(z)\mathrm{d}W_s, \quad \forall z \in \mathbb{D}, 0 \le t \le T$ (A3)
- $Y_t(z) \ge 0, \quad \forall z \in \mathbb{D}, 0 \le t \le T$ (A4)
- R_t *is an increasing process in* $[0, 1]$ *with* $R_0 = 0$ (A5)
- X, Y, Z, ξ, R *are* $\mathcal{F}^W$ *– adapted* (A6)
- $Z, \xi \in \mathbb{H}^2 := \left\{ \textit{progressively measurable } \varphi : \mathbb{E}\left[\int_0^T \varphi_s^2 \mathrm{d}s\right] < \infty \right\}.$ (A7)

Proof. $1 \implies 2:$

Take any randomised solution τ of (OptSEP') with external randomisation given by U, and define $R_t := \mathbb{E}\left[\mathbf{1}\{\tau \le t\} \middle| \mathcal{F}_t^W\right]$. Clearly R is an increasing, $\mathcal{F}_t^W$-measurable process in $[0, 1]$. From our assumptions on F we note that there exist $\mathcal{F}^W$-adapted X and $Z \in \mathbb{H}^2$ satisfying (A2), see for example Pardoux and Peng [1990, Proposition 2.2]. Also,

$$\int_t^T R_s \mathcal{L}F(W_s, s)\mathrm{d}s = \mathbb{E}^U\left[\int_{t\vee\tau}^T \mathcal{L}F(W_s, s)\mathrm{d}s\right],$$

so

$$\begin{aligned} X_0 &= \mathbb{E}\left[X_0 \middle| \mathcal{F}_0^W\right] \\ &= \mathbb{E}^W\left[\mathbb{E}^U\left[\int_\tau^T \mathcal{L}F(W_s, s)\mathrm{d}s + F(W_T, T)\right]\right] \\ &= \mathbb{E}\left[F(W_\tau, \tau) \middle| U\right] \\ &= x_0 \end{aligned}$$

For each $z \in \mathbb{D}$ let $\hat{Y}_t(z) := w_\mu(z) - \int_0^t (1 - R_s)\mathrm{d}L_s^z$ and note that

$$\begin{aligned} \int_0^t (1 - R_s)\mathrm{d}L_s^z &= \mathbb{E}^U\left[\int_0^{t\wedge\tau} \mathrm{d}L_s^z\right] \\ &= \mathbb{E}^U\left[L_{t\wedge\tau}^z\right], \end{aligned}$$

so $\mathbb{E}\left[\hat{Y}_T(z)\right] \ge 0$. Then there exists some $M_T(z)$ with $\mathbb{E}\left[M_T(z)\right] = 0$ such that $\hat{Y}_T(z) +$

$M_T(z) \geq 0$ almost surely. Choose $\xi(z)$ so that $\int_0^t \xi_s(z)\mathrm{d}W_s = \mathbb{E}\left[M_T(z)|\mathcal{F}_t^W\right]$ and define $Y_t(z) := \hat{Y}_t(z) + \int_0^t \xi_s(z)\mathrm{d}W_s$. Then, noting that $Y(z)$ is a supermartingale, we have $\mathcal{F}^W$-adapted Y and ξ satisfying (A3) and (A4). Furthermore, since $Y(z)$ is bounded for any $z \in \mathbb{D}$ we can choose $\xi(z) \in \mathbb{H}^2$.

$2 \implies 1$:

Take any $(X, Y(z), Z, \xi(z), R)$ satisfying (A1)-(A7) and let $\tau := \inf\{t \geq 0 : R_t \geq U\} \wedge T$ so that $R_t = \mathbb{E}^U\left[\mathbf{1}\{\tau \leq t\}\right]$. Then

$$\begin{aligned}
x_0 &= \mathbb{E}^W\left[F(W_T, T) - \int_0^T R_s \mathcal{L}F(W_s, s)\mathrm{d}s\right] \\
&= \mathbb{E}^W\left[F(W_T, T) - \int_0^T \int_0^1 \mathbf{1}\{\tau(u) \leq s\}\mathrm{d}u \mathcal{L}F(W_s, s)\mathrm{d}s\right] \\
&= \mathbb{E}^W\left[F(W_T, T) - \int_0^1 \int_{\tau(u)}^T \mathcal{L}F(W_s, s)\mathrm{d}s\mathrm{d}u\right] \\
&= \mathbb{E}^W\left[F(W_T, T) - \mathbb{E}^U\left[\int_\tau^T \mathcal{L}F(W_s, s)\mathrm{d}s\right]\right] \\
&= \mathbb{E}\left[F(W_\tau, \tau)\right].
\end{aligned}$$

Similarly, for any $z \in \mathbb{D}$,

$$\begin{aligned}
0 &\leq \mathbb{E}\left[Y_T(z)\right] \\
&= w_\mu(z) - \mathbb{E}\left[\int_0^T (1 - R_s)\mathrm{d}L_s^z\right] \\
&= w_\mu(z) - \mathbb{E}^W\left[\int_0^T \int_0^1 \mathbf{1}\{\tau(u) > s\}\mathrm{d}u\mathrm{d}L_s^z\right] \\
&= w_\mu(z) - \mathbb{E}^W\left[\int_0^1 \int_0^{\tau(u)} \mathrm{d}L_s^z\mathrm{d}u\right] \\
&= w_\mu(z) - \mathbb{E}\left[L_\tau^z\right].
\end{aligned}$$

In particular, $\mathbb{E}\left[L_\tau^{x_*}\right] = 0 = \mathbb{E}\left[L_\tau^{x^*}\right]$, so the stopping rule τ has absorbing barriers at x_* and x^*, and $(W_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable.

$\square$

Our optimisation problem is then to find

$$\sup_{(X,Y,Z,\xi,R)} \mathbb{E}\left[X_0\right] = \mathbb{E}\left[-\int_t^T R_s \mathcal{L}F(W_s,s)\mathrm{d}s + F(W_T,T)\right]$$

over solutions (X,Y,Z,ξ,R) to (A2)-(A6). Instead of requiring the positivity of $Y_t(z)$ at every z and time t, we can use a Lagrangian term to penalise $Y_t(z)<0$, and in fact it suffices to consider just $Y_T(z)$ since $Y_t(z)$ is a supermartingale for each z. Suppose now that $\operatorname{supp}(\mu)=\{z_k :\ k=1,\dots,n\}$ is finite. Our problem is then to find

$$\sup_{(Z,\xi,R)} \inf_{\substack{\Lambda\geq 0\\ \Lambda\in\mathcal{F}_T}} \left\{\mathbb{E}\left[\sum_k \Lambda(z_k)\left(w_\mu(z_k) - \int_0^T (1-R_s)\mathrm{d}L_s^{z_k} + \int_0^T \xi_s(z_k)\mathrm{d}W_s\right)\right] + \mathbb{E}\left[F(W_T,T) - \int_0^T R_s\mathcal{L}F(W_s,s)\mathrm{d}s\right]\right\},$$

where our supremum is over triples (Z,ξ,R) such that (A5) holds, R_t is adapted, and the processes $X^{R,Z}$ and $Y^{R,\xi}$ defined by (A2) and (A3) are also adapted. The infimum is over $\mathcal{F}_T$-measurable random variables $\Lambda(z)\geq 0$, for $z\in\mathbb{D}$.

Lemma 5.2. *It suffices to consider $\Lambda(x)\in\mathcal{F}_0$ for all x, i.e. constant Lagrange multipliers.*

Proof. Note that we can add an arbitrary $\mathcal{F}^W$-adapted process $\tilde{Y}_t$ to Y with $\mathbb{E}\left[\tilde{Y}_t\right]=0$ for all t by changing ξ to some $\tilde{\xi}$, so we can consider

$$Y_t^{R,\tilde{\xi}}(z) := Y_t^{R,\xi}(z) + \tilde{Y}_t = w_\mu(z) - \int_0^t (1-R_s)\mathrm{d}L_s^z + \int_0^t \xi_s(z)\mathrm{d}W_s + \tilde{Y}_t$$

for $0\leq t\leq T$.

If $Y_T^{R,\xi}(z)\geq 0$ almost surely for some $z\in\mathbb{D}$ then clearly

$$\inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_T} \mathbb{E}\left[\Lambda(z)Y_T^{R,\xi}(z)\right] = 0 = \inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_0} \mathbb{E}\left[\Lambda(z)Y_T^{R,\xi}(z)\right].$$

Suppose then that $Y_T^{R,\xi}<0$ with positive probability for some z. If $\mathbb{E}\left[Y_T^{R,\xi}\right]<0$ then

$$\inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_T} \mathbb{E}\left[\Lambda(z)Y_T^{R,\xi}(z)\right] = -\infty = \inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_0} \mathbb{E}\left[\Lambda(z)Y_T^{R,\xi}(z)\right].$$

If $\mathbb{E}\left[Y_T^{R,\xi}\right]\geq 0$ then we can choose $\tilde{Y}_t$ (and $\tilde{\xi}_t$) so that $Y_T^{R,\tilde{\xi}} = Y_T^{R,\xi} + \tilde{Y}_T \geq 0$ almost surely. In this case,

$$\begin{aligned}\inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_T}\mathbb{E}\left[\Lambda(z)\left(Y_T^{R,\tilde{\xi}}(z)+\tilde{Y}_T\right)\right] &= \inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_0}\mathbb{E}\left[\Lambda(z)\left(Y_T^{R,\tilde{\xi}}(z)+\tilde{Y}_T\right)\right]\\ &= \inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_0}\left\{\mathbb{E}\left[\Lambda(z)Y_T^{R,\xi}(z)\right]+\Lambda(z)\mathbb{E}\left[\tilde{Y}_T\right]\right\}\\ &= \inf_{\Lambda\geq 0,\Lambda\in\mathcal{F}_0}\mathbb{E}\left[\Lambda(z)Y_T^{R,\xi}(z)\right],\end{aligned}$$

so we have constructed a feasible solution with the same value. □

Suppose now that a minimax theorem applies, so we can swap the order of the infimum and supremum. Then,

$$\sup_{(Z,\xi,R)}\inf_{\Lambda\geq 0}\mathbb{E}\left[\sum_k\Lambda(z_k)Y_T^{R,\xi}-X_0^{R,Z}\right]=\inf_{\Lambda\geq 0}\sup_{(Z,\xi,R)}\left\{\sum_k\Lambda(z_k)\mathbb{E}\left[Y_T^{R,\xi}\right]-\mathbb{E}\left[X_0^{R,Z}\right]\right\},$$

where each $\Lambda(z)$ is now a constant, and we therefore wish to find

$$\begin{aligned}\inf_{\Lambda\geq 0}\sup_{(Z,\xi,R)}\Bigg\{&\mathbb{E}\left[F(W_T,T)-\int_0^T R_s\mathcal{L}F(W_s,s)\mathrm{d}s+\sum_k\Lambda(z_k)w_\mu(z_k)\right]\\ &-\mathbb{E}\left[\sum_k\Lambda(z_k)\int_0^T(1-R_s)\mathrm{d}L_s^{z_k}\right]\Bigg\}.\end{aligned}$$

We solve this problem and later show that the form of the optimal solution indeed implies that a minimax theorem holds, see Theorem 5.10.

In the above we can think of the process R as a control process, however we choose to rewrite the problem to put it in the usual form for a stochastic control problem. Define the control space $\mathcal{U}^d$ to be the set of progressively measurable, decreasing, predictable processes $(u_t)_{t\in[0,T]}$ with values in $[0,1]$ such that $u_T=0$, and for any control $u\in\mathcal{U}^d$ define a controlled process $X^{u,x,t}$ via

$$\mathrm{d}X_s^{u,x,t}=u_s\mathrm{d}W_s,\ s\geq t,\qquad X_t^{u,x,t}=x,$$

for each (x,t). We will drop the superscript dependence on the control, writing $X^{t,x}$, unless we wish to compare two controlled processes. The process $X^{t,x}$ then represents the stopped Brownian motion, with $u_s=1$ when the Brownian path is still running, and $u_s=0$ when it is completely stopped. Note that because we consider randomised stopping times we may have $u_s\in(0,1)$ in general, however in certain cases we can show that this is never optimal. For each (x,t) we also define a pairs of processes,

$(Y^{x,t}, Z^{x,t})$, to be the adapted solution, if one exists, of the BSDE

$$\begin{aligned}
\mathrm{d}Y_s^{x,t} &= k(u_s)\left(\frac{1}{2}u_s^2 F_{xx}(X_s^{x,t}, s) + F_t(X_s^{x,t}, s)\right)\mathrm{d}s + Z_s^{x,t}\mathrm{d}W_s, \quad t \le s \le T,\\
Y_T^{x,t} &= F(X_T^{x,t}, T),
\end{aligned}$$

where $k : [0,1] \to [0,1]$ is some function such that $k(0) = 1$ and $k(1) = 0$. We see that $k(u_s)$ represents R_s from the previous setup. Equivalently we can write

$$Y_s^{x,t} = F(X_T^{x,t}, T) - \int_s^T k(u_r)\left(\frac{1}{2}u_r^2 F_{xx}(X_r^{x,t}, r) + F_t(X_r^{x,t}, r)\right)\mathrm{d}r - \int_s^T Z_r^{x,t}\mathrm{d}W_r,$$

for $t \le s \le T$. It is well known, see for example Pham [2009, Theorem 6.2.1], that the above BSDE admits a unique solution $(Y^{x,t}, Z^{x,t})$ provided $F(X_T^{x,t}, T)$ is square integrable and we have certain Lipschitz continuity in the generator. For every (x,t) and $z \in \mathbb{D}$, we also define a pair of processes, $(\mathcal{Y}^{x,t}(z), \xi^{x,t}(z))$, to be the adapted solution of

$$\mathcal{Y}_s^{x,t}(z) = w_\mu(z) - \int_t^s \mathrm{d}L_r^x(X^{x,t}) + \int_t^s \xi_r^{x,t}\mathrm{d}W_r, \; s \ge t,$$

if one exists. Note that by Itô-Tanaka we can write this as

$$\mathcal{Y}_s^{x,t}(z) = w_\mu(z) - \left|X_s^{x,t} - z\right| + |x - z| + \int_t^s \tilde{\xi}_r^{x,t}\mathrm{d}W_r = \int_t^s \tilde{\xi}_r^{x,t}\mathrm{d}W_r - \left(U_\mu(z) + \left|X_s^{x,t} - z\right|\right),$$

for some $\tilde{\xi}$. Suppose that $W_0 = 0$ and denote $X_s := X_s^{0,0}$, and similarly $Y, Z, \mathcal{Y}, \xi$. Then, we can rewrite (OptSEP') as

$$\sup_{u \in \mathcal{U}} \mathbb{E}\left[Y_0\right] = \mathbb{E}\left[F(X_T, T) - \int_0^T k(u_s)\left(\frac{1}{2}u_s^2 F_{xx}(X_s, s) + F_t(X_s, s)\right)\mathrm{d}s\right]$$

subject to

- $\mathcal{Y}_t(z) \ge 0, \quad \forall 0 \le t \le T, z \in \mathbb{D}.$

Once again we incorporate this condition into our objective function using Lagrange multipliers to get an unconstrained problem:

$$\sup_{u \in \mathcal{U}} \inf_{\Lambda \ge 0} \mathbb{E}\left[F(X_T, T) - \int_0^T k(u_s)\left(\frac{1}{2}u_s^2 F_{xx}(X_s, s) + F_t(X_s, s)\right)\mathrm{d}s + \sum_{z \in \mathrm{supp}(\mu)} \Lambda(z)\mathcal{Y}_T(z)\right].$$

We solve the dual problem,

$$\inf_{\Lambda\geq 0}\sup_{u\in\mathcal{U}}\left\{\mathbb{E}\left[F(X_T,T)-\int_0^T k(u_s)\left(\frac{1}{2}u_s^2F_{xx}(X_s,s)+F_t(X_s,s)\right)\mathrm{d}s\right]\right.$$
$$\left.-\mathbb{E}\left[\sum_k\lambda_k\left(U_\mu(z_k)+|X_T-z_k|\right)\right]\right\},$$

where $\lambda_k:=\Lambda(z_k)$ are non-negative constants, and show in Theorem 5.10 that the values are equal.

The supremum part of this optimisation is now in the standard form for a stochastic optimal control problem. However, to use standard results in the field, such as the stochastic maximum principle, we require some conditions on our objective function. We summarise these conditions and the resulting maximum principles in the next section.

5.2 Stochastic Optimal Control Summary

We follow the approach of Yong and Zhou [1999, Chapter 3], adapting the results to fit our problem more clearly. Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ be a given filtered probability space on which a standard Brownian motion W is given, and $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated by W, augmented by the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the controlled process X given by

$$\mathrm{d}X_t=u_t\mathrm{d}W_t,\ t\geq 0,\quad X_0=x_0,$$

for control processes u_t with $u_t\in U$ for each $t\in[0,T]$, for some bounded set $U\subseteq\mathbb{R}$. The set of feasible controls is $\mathcal{U}:=\{u:[0,T]\to U:\ u \text{ is progressively measurable}\}$, and we have cost functional

$$J(u_\cdot)=\mathbb{E}\left[\int_0^T f(X_t,t,u_t)\mathrm{d}t+h(X_T)\right]$$

for $f:\mathbb{R}\times[0,T]\times U\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$. For their analysis, Yong and Zhou require the following assumptions:

(B1) f,h are measurable, and there exist a constant $L>0$ and a modulus of continuity $\bar{\omega}:[0,\infty)\to[0,\infty)$ such that for all $x,\hat{x},t,u,\hat{u}$, and any $\phi(x,t,u)=$

$f(x,t,u), h(x)$,

- $|\phi(x,t,u) - \phi(\hat{x},t,\hat{u})| \leq L|x-\hat{x}| + \bar{\omega}(|u-\hat{u}|)$,
- $|\phi(0,t,u)| \leq L$

(B2) f, h are C^2 in x, and there exist a constant $L > 0$ and a modulus of continuity $\bar{\omega}$: $[0,\infty) \to [0,\infty)$ such that for all $x, \hat{x}, t, u, \hat{u}$, and any $\phi(x,t,u) = f(x,t,u), h(x)$,

- $|\phi_x(x,t,u) - \phi_x(\hat{x},t,\hat{u})| \leq L|x-\hat{x}| + \bar{\omega}(|u-\hat{u}|)$,
- $|\phi_{xx}(x,t,u) - \phi_{xx}(\hat{x},t,\hat{u})| \leq \bar{\omega}(|x-\hat{x}| + |u-\hat{u}|)$.

The problem is then to find

$$\sup_{u\in\mathcal{U}} J(u.)$$

and assess the existence of an optimal control, i.e. does there exist some $\bar{u} \in \mathcal{U}$ such that $J(\bar{u}.) = \inf_{u\in\mathcal{U}} J(u.)$. If $\bar{u}$ is an optimal control, with corresponding process $\bar{X}$, we call $(\bar{X}, \bar{u})$ an optimal pair.

Define the Hamiltonian, H, and the generalised Hamiltonian, G, to be the functions $H, G : \mathbb{R} \times [0,T] \times U \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} H(x,t,u,p,q) &= qu + f(x,t,u), \\ G(x,t,u,p,P) &= \frac{1}{2}u^2 P + f(t,x,u). \end{aligned}$$

For a feasible pair $(\bar{X}, \bar{u})$ the first-order adjoint equation is the BSDE

$$\mathrm{d}p_t = -f_x(\bar{X}_t, t, \bar{u}_t)dt + q_t \mathrm{d}W_t,\ t \in [0,T], \quad p_T = h_x(\bar{X}_T).$$

With the assumptions (B1), (B2), this has a unique adapted solution (p,q) for any $(\bar{X}, \bar{u}) \in L^2 \times \mathcal{U}$. We call p the first-order adjoint process and it represents the marginal value of an increase in X. For a pair $(\bar{X}, \bar{u})$ and their corresponding (p_t, q_t), the second-order adjoint equation is given by

$$\mathrm{d}P_t = -H_{xx}(\bar{X}_t, t, \bar{u}_t, p_t, q_t)\mathrm{d}t + Q_t \mathrm{d}W_t,\ t \in [0,T], \quad P_T = h_{xx}(\bar{X}_t).$$

As above, this has a unique adapted solution (P,Q) under our earlier assumptions (B1) and (B2).

If $(\bar{X}, \bar{u})$ is a feasible pair with corresponding adjoints (p,q,P,Q), then $(\bar{X}, \bar{u}, p, q, P, Q)$

is a feasible 6-tuple, if $(\bar{X}, \bar{u})$ is an optimal pair then we call the 6-tuple optimal also. For any feasible 6-tuple $(\bar{X}, \bar{u}, p, q, P, Q)$ we can define a $\mathcal{H}$-function given by

$$\begin{aligned}\mathcal{H}(x,t,u) &= H(x,t,u,p_t,q_t) + \frac{1}{2}u^2 P_t - u\bar{u}_t P_t \\ &= \frac{1}{2}u^2 P_t - u\bar{u}_t P_t + f(x,t,u) + uq_t.\end{aligned}$$

Theorem 5.3 (Stochastic Maximum Principle, Yong and Zhou [1999] Theorem 3.2)**.** *Suppose that (B1) and (B2) hold and let $(\bar{X}, \bar{u})$ be an optimal pair. Then there are pairs of processes $(p,q), (P,Q) \in L^2 \times L^2$ satisfying the first and second-order adjoint equations, respectively, such that*

$$H(\bar{X}_t, t, \bar{u}_t, p_t, q_t) - H(\bar{X}_t, t, u, p_t, q_t) - \frac{1}{2}(\bar{u}_t - u)^2 P_t \geq 0, \tag{SMP1}$$
$$\forall u \in U, \;\; a.e. \;\; t \in [0,T], \; \mathbb{P} - a.s.$$

or equivalently,

$$\mathcal{H}(\bar{X}_t, t, \bar{u}_t) = \sup_{u \in U} \mathcal{H}(\bar{X}_t, t, u), \qquad a.e. \;\; t \in [0,T], \; \mathbb{P} - a.s. \tag{SMP2}$$

Corollary 5.4. *If U is convex, and coefficients are C^1 in u, then* (SMP1) *implies*

$$H_u(\bar{X}_t, t, \bar{u}_t, p_t, q_t)(u - \bar{u}_t) \leq 0, \qquad \forall u \in U, \;\; a.e. \;\; t \in [0,T], \; \mathbb{P} - a.s. \tag{SMP3}$$

Theorem 5.5 (Yong and Zhou [1999] Theorem 5.2)**.** *Suppose (B1) and (B2) hold and suppose further that U is convex, f is locally Lipschitz in u, f_x is continuous in (x,u), and h is convex. Let $(\bar{X}, \bar{u}, p, q, P, Q)$ be a feasible 6-tuple. Suppose that $H(\cdot, t, \cdot, p_t, q_t)$ is concave for all $t \in [0,T]$ almost surely, and*

$$\mathcal{H}(\bar{X}_t, t, \bar{u}_t) = \sup_{u \in U} \mathcal{H}(\bar{X}_t, t, u), \qquad a.e. \;\; t \in [0,T], \; \mathbb{P} - a.s.$$

Then $(\bar{X}, \bar{u})$ is an optimal pair.

5.3 Stochastic Maximum Principle for our Problem

To apply the results of the last section to our problem, we must choose f and h to satisfy $(A1)$ and $(A2)$. Our problem is currently written in a form which suggests that the natural choices of f and h are $\tilde{f}(x,t,u) = -k(u)\left(\frac{1}{2}u_s^2 F_{xx}(x,t) + F_t(x,t)\right)$ and

$\tilde{h}(x) = F(x,T) - \sum_k \lambda_k \left(U_\mu(z_k) + |x - z_k|\right)$, respectively.

We restrict ourselves to payoff functions F such that $f = \tilde{f}$ satisfies (B1) and (B2). Note in particular that if F depends only on time, then $f(x,t,u) = k(u)F'(t)$ trivially satisfies (B2), and satisfies (B1) for any Lipschitz k provided F has bounded derivative on $[0,T]$. We will suppose initially that $F : [0,T] \to \mathbb{R}$ is a C^1 function of time only and is concave, increasing with $F(0) = 0$.

We have not yet specified the function k beyond that it is some function $k : [0,1] \to [0,1]$ such that $k(0) = 1$ and $k(1) = 0$. Some natural choices therefore include $k(u) = \mathbf{1}\{u = 0\}$, $k(u) = 1 - u^a$, and $k(u) = (1-u)^a$, for $a > 0$. We choose $k(u) = 1 - u^2$ as this provides suitable regularity (Lipschitz, continuously differentiable) and also interesting maximum principle results in the case of the Root embedding. For other embeddings it may be necessary to choose a different function k.

Assumption. *Suppose that μ is a probability measure with finitely many atoms at $\{x_* = z_0, z_1, \dots, z_n = x^*\}$. We consider payoff functions $F \in C^1([0,T])$ that are concave, increasing functions of time with $F(0) = 0$. We also fix $k(u) := (1-u)^2$ for $u \in [0,1]$.*

For h we have the obvious problem that, due to the moduli, $\tilde{h}$ is not differentiable, and so we approximate $\tilde{h}$ by a smooth function. For any $\varepsilon > 0$ let $h^\varepsilon : \mathbb{R} \to \mathbb{R}$ be a convex, C^2, even function such that $h^\varepsilon(x) = |x|$ for $|x| \geq \varepsilon$, $h^\varepsilon \geq |x|$ for $|x| < \varepsilon$, and $h^\varepsilon_{xx}(x)$ is maximised at $x = 0$. Then we let $h(x) = \sum_k \lambda_k \left(U_\mu(z_k) + h^\varepsilon(x - z_k)\right) - F(x,T)$. We consider $\varepsilon < \min_{j,k}\{|z_j - z_k|\}$ so that the sets $\operatorname{supp}(h^\varepsilon(\cdot - z_k))$ are disjoint.

Finally then, our problem is to find

$$\inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}^d} J(u,\lambda) := \mathbb{E}\left[\int_0^T f(t,u_t)\mathrm{d}t + h(X_T)\right], \tag{OptCon}$$

where

- $\mathcal{U}^d := \{u : [0,T] \times \Omega \to [0,1] \big|\, u \text{ is progressively measurable, decreasing, } u_T = 0\}$
- $\Lambda := \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n : \ \lambda_k \geq 0 \ \forall k\}$
- $f(t,u) := -(1-u^2)F'(t), \qquad \forall t \in [0,T], u \in [0,1]$
- $h(x) := F(T) - \sum_k \lambda_k \left(U_\mu(z_k) + h^\varepsilon(x - z_k)\right), \qquad \forall x \in \mathbb{R}$
- $X_t = \int_0^t u_s \mathrm{d}W_s, \qquad \forall t \in [0,T], u_t \in \mathcal{U}^d.$

The results of Yong and Zhou [1999] hold when we consider the above optimisation with controls taken in $\mathcal{U} := \{u : [0,T] \times \Omega \to [0,1] : \ u \text{ is progressively measurable}\}$. For now we apply the results to $\mathcal{U}^d \subseteq \mathcal{U}$, and later show that the problems are in fact equivalent.

Suppose that we have an optimal pair $(\bar{X}, \bar{u})$ for this problem. The Hamiltonians and adjoint equations from the previous section become

- $H(x,t,u,p,q) := qu - (1-u^2)F'(t)$,
- $G(x,t,u,p,P) := \dfrac{1}{2}u^2P - (1-u^2)F'(t)$,
- $\mathrm{d}p_t = q_t \mathrm{d}W_t, \quad p_T = -\sum_k \lambda_k h_x^\varepsilon(\bar{X}_T - z_k)$,
- $\mathrm{d}P_t = Q_t \mathrm{d}W_t, \quad P_T = -\sum_k \lambda_k h_{xx}^\varepsilon(\bar{X}_T - z_k)$,
- $\mathcal{H}(x,t,u) := \dfrac{1}{2}u^2P_t - u\bar{u}_tP_t - (1-u^2)F'(t) + uq_t$.

Since our functions satisfy (B1) and (B2), we know that for any feasible pair $(\bar{X}, \bar{u})$, the above adjoint equations have unique, adapted solutions.

The stochastic maximum principles imply the following necessary conditions on our optimal pair:

$$\begin{aligned}
&\text{(SMP1)} \implies (\bar{u}_t - u)\left(q_t + F'(t)(\bar{u}_t + u) - \frac{1}{2}(\bar{u}_t - u)P_t\right) \geq 0, \\
&\text{(SMP2)} \implies -\frac{1}{2}\bar{u}_t^2P_t - (1-\bar{u}_t^2)F'(t) + \bar{u}_tq_t = \sup_{u\in[0,1]}\left\{\frac{1}{2}u^2P_t + u(q_t - \bar{u}_tP_t)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. -(1-u^2)F'(t)\right\}, \\
&\text{(SMP3)} \implies (q_t + 2\bar{u}_tF'(t))(u - \bar{u}_t) \leq 0,
\end{aligned}$$

for all $u \in [0,1]$ and $t \in [0,T]$.

We can see immediately from the adjoint equations that p_t and P_t are local martingales with $p_T = -\sum_k \lambda_k h_x^\varepsilon(x - z_k)$ and $P_T = -\sum_k \lambda_k h_{xx}^\varepsilon(x - z_k)$. Therefore we must have

$$\begin{aligned}
p_t &= \mathbb{E}\left[-\sum_k \lambda_k h_x^\varepsilon(\bar{X}_T - z_k)\middle|\mathcal{F}_t\right], \\
P_t &= \mathbb{E}\left[-\sum_k \lambda_k h_{xx}^\varepsilon(\bar{X}_T - z_k)\middle|\mathcal{F}_t\right].
\end{aligned}$$

We can see that Q doesn't appear in any of the maximum principles for any problem, however we do require q. By the Clark-Ocone formula, see Rogers and Williams [1994, Theorem 41.9], we have that

$$\mathrm{d}p_t = \mathbb{E}\left[-\sum_k \lambda_k h^\varepsilon_{xx}(\bar{X}_T - z_k) \middle| \mathcal{F}_t \right] \mathrm{d}\bar{X}_t = P_t \bar{u}_t \mathrm{d}t,$$

so, comparing with the first-order adjoint equation, we must have

$$q_t = P_t \bar{u}_t.$$

Substituting this into our maximum principle gives

$$\text{(SMP1)} \implies (\bar{u}_t^2 - u^2)\left(\frac{1}{2}P_t + F'(t)\right) \geq 0, \quad \forall u \in [0,1], t \in [0,T].$$

In particular we must have

$$\frac{1}{2}P_t + F'(t) > 0 \implies \bar{u}_t = 1 \tag{5.1}$$

$$\frac{1}{2}P_t + F'(t) < 0 \implies \bar{u}_t = 0 \tag{5.2}$$

$$\bar{u}_t = 1 \implies \frac{1}{2}P_t + F'(t) \geq 0$$

$$\bar{u}_t = 0 \implies \frac{1}{2}P_t + F'(t) \leq 0.$$

At first it appears as if these completely determine our optimal control, and therefore our stopping rule, by forcing $\bar{u}_t = \mathbf{1}\{\frac{1}{2}P_t + F'(t) > 0\}$, where we could take a weak inequality in the indicator as we have some freedom at times t where $\frac{1}{2}P_t + F'(t) = 0$. However, note that then we have a control $\bar{u}_t$ which is determined by $P_t = \mathbb{E}\left[-\sum_k \lambda_k h^\varepsilon_{xx}(\bar{X}_T - z_k) \middle| \mathcal{F}_t \right]$, which depends on $\bar{u}_t$ through $\bar{X}_T$. We therefore have a circular argument and need to think further.

Consider running our Brownian motion initially up to some time t, so $\bar{u}_s = 1$ and $\bar{X}_s = W_s$ for $s < t$. If $\bar{u}_t = 0$ then we stop the process immediately and we have that

$\bar{X}_s = W_t$ for all $s \in [t, T]$. Then, we know that

$$\begin{aligned}\bar{u}_t = 0 \implies F'(t) \leq -\frac{1}{2}P_t &= \frac{1}{2}\mathbb{E}\left[\sum_k \lambda_k h^\varepsilon_{xx}(\bar{X}_T - z_k)\Big|\mathcal{F}_t\right] \\ &= \frac{1}{2}\mathbb{E}\left[\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k)\Big|\mathcal{F}_t\right] \\ &= \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k).\end{aligned}$$

In particular, $\bar{u}_t > 0$ for $t < \inf\{s \geq 0 : \; F'(t) - \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k) \leq 0\}$.

Up to now we have considered randomised stopping time solutions of our problem by allowing $u_t \in (0,1)$ for $0 < t < T$. However, we expect the solution of this Root embedding-type problem to be a true stopping time, so that $u_t \in \{0,1\}$ for any t. Define

$$\bar{\mathcal{U}} := \{u : [0,T] \times \Omega \to \{0,1\} : \; u \text{ is progressively measurable, decreasing, } u_T = 0\},$$

so for any $u \in \bar{\mathcal{U}}$ there is a stopping time τ for W such that $u_t := \mathbf{1}\{t < \tau\}$. We now show that any optimiser lies in $\bar{\mathcal{U}} \subseteq \mathcal{U}^d \subseteq \mathcal{U}$, and therefore that the results of Yong and Zhou [1999] hold in (OptCon), where we can without loss of generality optimise over $u \in \bar{\mathcal{U}}$.

Theorem 5.6. *For any $\lambda \in \Lambda$ and $u \in \mathcal{U}$, there exists $\bar{u} \in \bar{\mathcal{U}}$ such that $J(u,\lambda) \leq J(\bar{u},\lambda)$, and so*

$$\sup_{u \in \mathcal{U}} J(u,\lambda) = \sup_{u \in \mathcal{U}^d} J(u,\lambda) = \sup_{u \in \bar{\mathcal{U}}} J(u,\lambda).$$

Proof. Fix any $\lambda \in \Lambda$ and $u \in \mathcal{U}$ with corresponding controlled process X. We construct $\tilde{u}$, $\tilde{X}$ where $\tilde{u} \in \bar{\mathcal{U}}$ and $J(u,\lambda) \leq J(\tilde{u},\lambda)$. Let $\tau_t := \inf\left\{s \geq 0 : \; \int_0^t u_s^2 \mathrm{d}s \geq t\right\} \wedge T$ and define a new Brownian motion B by $B_t := X_{\tau_t}$ for $t \in [0,T]$. Then we define $\tilde{X}_t := \int_0^t \tilde{u}_s \mathrm{d}B_s$ where $\tilde{u}_t := \mathbf{1}\{t < \sigma_T\}$ for $\sigma_T := \int_0^T u_s^2 \mathrm{d}s$. Note that

$$\tilde{X}_T = B_{T \wedge \sigma_T} = B_{\sigma_T} = X_{\tau_{\sigma_T}} = X_T$$

since for $t > \tau_{\sigma_T}$ we must have $u_t = 0$. Then $h(\tilde{X}_T) = h(X_T)$ and it remains to show that $-\int_0^T(1-\tilde{u}_t^2)F'(t)\mathrm{d}t \geq -\int_0^T(1-u_t^2)F'(t)\mathrm{d}t$. We can write

$$\begin{aligned}\int_0^T \tilde{u}_t^2 F'(t)\mathrm{d}t &= \int_0^{\sigma_T} F'(t)\mathrm{d}t \\ &= F(\sigma_T) \\ &= F\left(\int_0^T u_t^2 \mathrm{d}t\right),\end{aligned}$$

so it suffices to show that $F\left(\int_0^T u_t^2\mathrm{d}t\right) \geq \int_0^T u_t^2 F'(t)\mathrm{d}t$. Let $a(t) := F\left(\int_0^t u_s^2\mathrm{d}s\right)$ and $b(t) := \int_0^t u_s^2 F'(s)\mathrm{d}s$ for $t \in [0,T]$. Then $a(0) = 0 = b(0)$, but

$$a'(t) = u_t^2 F'\left(\int_0^t u_s^2\mathrm{d}s\right) \geq u_t^2 F'(t) = b'(t)$$

since $\int_0^t u_s^2\mathrm{d}s \leq t$ and F' is decreasing. Therefore we have $a(T) \geq b(T)$. □

Thus far we have not proved the existence of an optimal control for any $\lambda \in \Lambda$, and although Yong and Zhou [1999, Theorem 5.2] provides such a proof, this result requires h to be convex and so cannot be applied to our problem. Instead we use the previous result, noting that there is a bijection between $u \in \bar{\mathcal{U}}$ and stopping times $\tau \leq T$ given by $u_t = \mathbf{1}\{t < \tau\}$. Note then that

$$\begin{aligned}J(u,\lambda) = \hat{J}(\tau,\lambda) &:= \mathbb{E}\left[F(\tau)\right] - \sum_k \lambda_k \left(U_\mu(z_k) + \mathbb{E}\left[h^\varepsilon(W_\tau - z_k)\right]\right) \\ &= \mathbb{E}\left[\int_0^\tau \left(F'(t) - \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k)\right)\mathrm{d}t\right] \\ &\qquad - \sum_k \lambda_k \left(U_\mu(z_k) + h^\varepsilon(W_0 - z_k)\right).\end{aligned}$$

Corollary 5.7. *For any $\lambda \in \Lambda$ there exists an optimal control u^λ corresponding to some stopping time τ^λ.*

Proof. For a given λ, $\sup_{u\in\bar{\mathcal{U}}} J(u,\lambda) = \sup_{\tau\leq T} \hat{J}(\tau,\lambda)$, but by Prokhorov's theorem the set of stopping times τ such that $\tau \leq T$ is sequentially compact (since it is closed), so there is an optimiser. □

The above results imply that for any $\lambda \in \Lambda$, there is a stopping time τ^λ and $u^{\tau^\lambda} \in \bar{\mathcal{U}}$ such that $J(u^{\tau^\lambda},\lambda) = \sup_{u\in\mathcal{U}^d} J(u,\lambda)$, where u^{τ^λ} satisfies (SMP1) and $u_t^{\tau^\lambda} = \mathbf{1}\{t < \tau^\lambda\}$. From here on we will denote these maximisers by τ^λ and u^{τ^λ}.

From (SMP1) we see that for $t < \tau^\lambda$

$$F'(t) > \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k) \implies u_t^{\tau^\lambda} > 0 \implies u_t^{\tau^\lambda} = 1,$$

so any feasible stopping time must satisfy $\tau^\lambda \geq \inf\{t \geq 0 : \; F'(t) \leq \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k)\} \wedge T$ and in fact $F'(\tau^\lambda) \leq \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_{\tau^\lambda} - z_k)$.

With this in mind, for $\lambda \in \Lambda$ let

$$\sigma^\lambda := \inf\left\{t \geq 0 : \; (W_t, t) \notin \mathcal{D}^\lambda\right\},$$
$$\mathcal{D}^\lambda := \left\{(x,t) : \; F'(t) > \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(x - z_k), \; t < T\right\}$$

Note that $F'(\sigma^\lambda) = \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_{\sigma^\lambda} - z_k)$ on $\{\sigma^\lambda < T\}$, since F' and h^ε_{xx} are continuous.

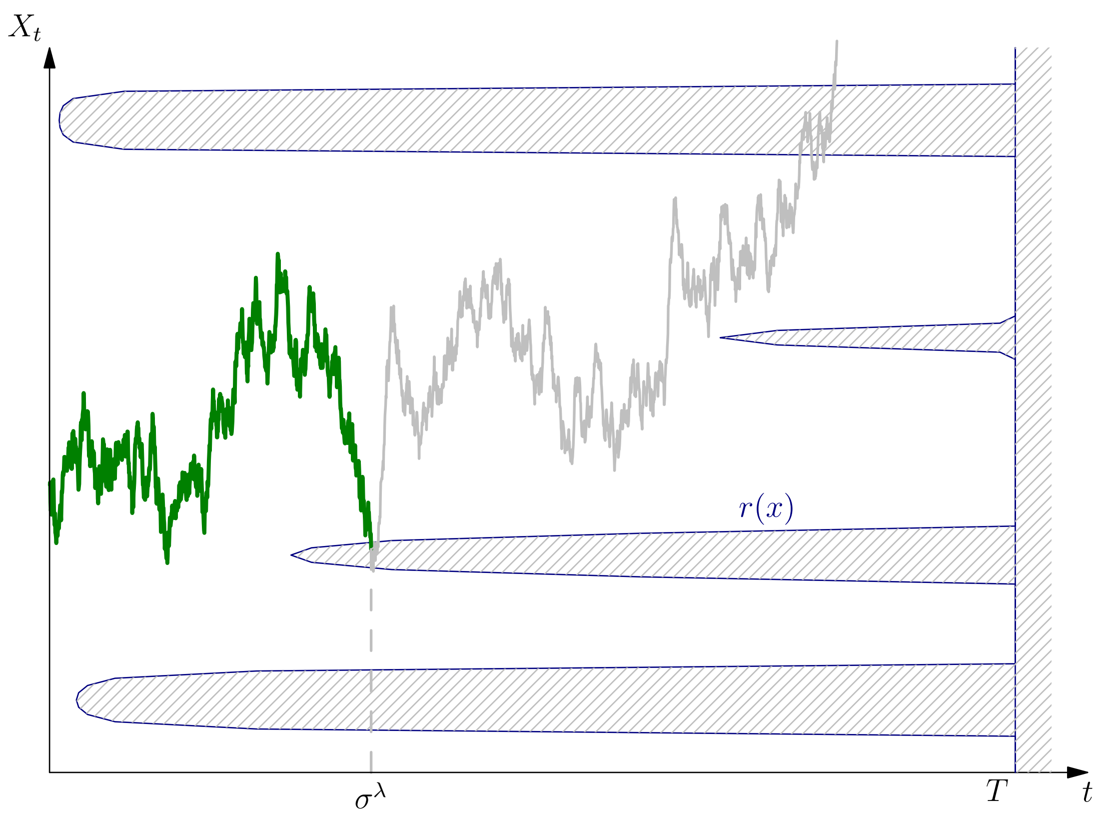

Figure 5-1: An example stopping region corresponding to σ^λ for some λ.

Since we chose F to be concave, F' is decreasing and the stopping region of σ^λ is therefore a Root barrier. We can easily check that this proposed control gives a feasible 6-tuple that satisfies the stochastic maximum principle, but we do not necessarily have optimality. Note that Theorem 5.5 (Yong and Zhou [1999, Theorem 5.2]) does not apply here since we are considering increasing F and therefore H is not concave. However, we can prove using the stochastic maximum principle as follows.

Theorem 5.8. *For any $\lambda \in \Lambda$, the 6-tuple $\left(X, u^{\sigma^\lambda}, p, q, P, Q\right)$ generated by $u_t^{\sigma^\lambda} = \mathbf{1}\left\{t < \sigma^\lambda\right\}$, where*

$$\sigma^\lambda := \inf\left\{t \geq 0 : \ F'(t) = \frac{1}{2}\sum_k \lambda_k h_{xx}^\varepsilon(W_t - z_k)\right\} \wedge T,$$

is optimal in the set of decreasing controls.

Proof. We know by Corollary 5.7 that an optimal $u \in \bar{\mathcal{U}}$ exists. Let the corresponding 6-tuple be $(X, u, p(u), q(u), P(u), Q(u))$, and let τ be the stopping time associated to u. We have already seen that the stochastic maximum principle implies that $\tau \geq \sigma^\lambda$ and $F'(\tau) \leq \frac{1}{2}\sum_k \lambda_k h_{xx}^\varepsilon(W_\tau - z_k)$ almost surely.

For any z_k take an interval $(a, b) \subseteq (z_k - \varepsilon, z_k + \varepsilon)$ such that $\mathbb{P}\left(W_{\sigma^\lambda} \in (a, b)\right) > 0$ and consider the paths of the Brownian motion for $t \geq \sigma^\lambda$. There is a positive probability that the process remains in the interval (a, b) up to time T, so $\mathbb{P}\left(W_t \in (a, b)\, \forall t \in [\sigma^\lambda, T]\right) > 0$. In particular, since F is concave, on this set we have that $F'(t) < -\frac{1}{2}\sum_k \lambda_k h_{xx}^\varepsilon(\bar{X}_T - z_k)$.

Suppose that $\tau > \sigma^\lambda$ for some $\omega \in \{W_t \in (a, b)\, \forall t \in [\sigma^\lambda, T]\}$, so $u_{\sigma^\lambda} = 1$. Then

$$\begin{aligned} -\frac{1}{2}P_{\sigma^\lambda} &= \mathbb{E}\left[\frac{1}{2}\sum_k \lambda_k h_{xx}^\varepsilon(\bar{X}_T - z_k)\Big|\mathcal{F}_{\sigma^\lambda}\right] \\ &> F'(\sigma^\lambda), \end{aligned}$$

but by (5.2) we must then have $u_{\sigma^\lambda} = 0$, which is a contradiction. Then for any $\omega \in \{W_{\sigma^\lambda} \in (a, b)\, \forall t \in [\sigma^\lambda, T]\}$ we must have $u_{\sigma^\lambda}(\omega) = 0$. However, since u is progressively measurable, any stopping rule can not distinguish between paths which will remain in the interval (a, b) and those that will leave this region. Therefore we must have $u_{\sigma^\lambda}(\omega) = 0$ for all $\omega \in \{W_{\sigma^\lambda} \in (a, b)\}$. $\square$

5.4 Optimal Lagrange Multipliers

We now know that (OptCon) is equivalent to

$$\inf_{\lambda\in\Lambda} \bar{J}(\lambda) := J(u^{\sigma^\lambda}, \lambda) = \mathbb{E}\left[F(\sigma^\lambda)\right] - \sum_k \lambda_k \mathbb{E}\left[U_\mu(z_k) + h^\varepsilon(W_{\sigma^\lambda} - z_k)\right],$$

where

$$\sigma^\lambda := \inf\left\{t \geq 0 :\ (W_t, t) \notin \mathcal{D}^\lambda\right\},$$
$$\mathcal{D}^\lambda := \left\{(x,t) :\ F'(t) > \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(x - z_k),\ t < T\right\}.$$

For the existence of optimisers note that we can restrict Λ to a compact subset of $\mathbb{R}^n$, since for any j we can without loss of generality consider only $\lambda_j \in [\underline{\lambda}, \bar{\lambda}]$, where $\bar{\lambda}$ and $\underline{\lambda}$ are such that

$$F'(0) = \frac{1}{2}\bar{\lambda} h^\varepsilon_{xx}(0), \quad F'(T) = \frac{1}{2}\underline{\lambda} h^\varepsilon_{xx}(0).$$

With this value of λ_j, the atom of the optimal stopping region at z_j reaches $t = 0$, i.e. $(z_j, 0) \in \left(\mathcal{D}^\lambda\right)^\complement$, regardless of the values of λ_k for $k \neq j$. Therefore we can consider our optimisation over the compact set $\bar{\Lambda} := [\underline{\lambda}, \bar{\lambda}]^n \subseteq \mathbb{R}^n$.

We now show that the optimal λ is such that the stopped Brownian motion has the correct local time at the atoms of μ, i.e. $U_\mu(z_k) + \mathbb{E}\left[h^\varepsilon(W_{\tau^\lambda} - z_k)\right] = 0$ for all k. To see why this may be true, note that if we have sufficient regularity on τ^λ for any λ, then for any j

$$\begin{aligned}
\partial_{\lambda_j}\bar{J}(\lambda) &= \partial_{\lambda_j}\mathbb{E}\left[\int_0^{\tau^\lambda}\left(F'(t) - \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k)\right)\mathrm{d}t\right] - \left(U_\mu(z_j) + h^\varepsilon(X_0 - z_j)\right)\\
&\approx \mathbb{E}\left[\left(F'(\tau^\lambda) - \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_{\tau^\lambda} - z_k)\right)\partial_{\lambda_j}\tau^\lambda - \int_0^{\tau^\lambda}\frac{1}{2}h^\varepsilon_{xx}(W_t - z_j)\mathrm{d}t\right]\\
&\qquad - \left(U_\mu(z_j) + h^\varepsilon(X_0 - z_j)\right)\\
&= -\mathbb{E}\left[\int_0^{\tau^\lambda}\frac{1}{2}h^\varepsilon_{xx}(W_t - z_j)\mathrm{d}t\right] - \left(U_\mu(z_j) + h^\varepsilon(X_0 - z_j)\right)\\
&= -U_\mu(z_j) - \mathbb{E}\left[h^\varepsilon(W_{\tau^\lambda} - z_j)\right].
\end{aligned}$$

Provided the objective function is concave in λ, we have our minimiser at the point where $\partial_{\lambda_j}\bar{J}(\lambda) = 0$ for each j. The technicalities here include making sense of the derivative $\partial_{\lambda_j}\tau^\lambda$ and we give an alternative proof.

Theorem 5.9. *The problem* (OptCon) *has optimisers, and*

$$J\left(u^{\sigma^{\hat{\lambda}}}, \hat{\lambda}\right) = \inf_{\lambda\in\Lambda} \sup_{u\in\bar{\mathcal{U}}} J(u,\lambda),$$

where $\hat{\lambda} \in \Lambda$ *is such that*

$$U_\mu(z_k) + \mathbb{E}\left[h^\varepsilon(W_{\sigma^{\hat{\lambda}}} - z_k)\right] = 0 \quad \forall k.$$

Proof. Since we optimise over the compact set $\bar{\Lambda}$, we know some optimiser λ^* exists. Also, note that for sufficiently large T, we can always find $\lambda \in \Lambda$ satisfying the above condition.

For any $\lambda \in \Lambda$,

$$\bar{J}(\lambda) := J(u^{\sigma^\lambda}, \lambda) = \mathbb{E}\left[F(\sigma^\lambda)\right] - \sum_k \lambda_k \mathbb{E}\left[U_\mu(z_k) + h^\varepsilon(W_{\sigma^\lambda} - z_k)\right],$$

and therefore by our choice of $\hat{\lambda}$ we have

$$J(u^{\sigma^{\hat{\lambda}}}, \lambda) = \mathbb{E}\left[F(\sigma^{\hat{\lambda}})\right] = J(u^{\sigma^{\hat{\lambda}}}, \hat{\lambda}) =: \bar{J}(\hat{\lambda}).$$

Since $\bar{J}(\lambda) := J(u^{\sigma^\lambda}, \lambda) = \sup_{u\in\bar{\mathcal{U}}} J(u,\lambda)$, we must have $\bar{J}(\lambda) \geq J(u^{\sigma^{\hat{\lambda}}}, \hat{\lambda}) = \bar{J}(\hat{\lambda})$ for any $\lambda \in \Lambda$. Then $\hat{\lambda}$ is an optimiser. □

Now we have the optimising pair $(u^{\sigma^{\hat{\lambda}}}, \hat{\lambda})$, we can prove that we were justified in swapping the order of the supremum and infimum to obtain (OptCon).

Theorem 5.10. *There is no duality gap in the sense that*

$$\sup_{u\in\mathcal{U}} \inf_{\lambda\in\Lambda} J(u,\lambda) = \inf_{\lambda\in\Lambda} \sup_{u\in\mathcal{U}} J(u,\lambda).$$

Proof. It is well know that

$$\sup_{u\in\mathcal{U}} \inf_{\lambda\in\Lambda} J(u,\lambda) \leq \inf_{\lambda\in\Lambda} \sup_{u\in\mathcal{U}} J(u,\lambda),$$

since $J(u,\lambda) \leq \sup_{u\in\mathcal{U}} J(u,\lambda)$ for any $u \in \mathcal{U}$, $\lambda \in \Lambda$, and therefore $\inf_{\lambda\in\Lambda} J(u,\lambda) \leq \inf_{\lambda\in\Lambda} \sup_{u\in\mathcal{U}} J(u,\lambda)$ for any u.

We have shown in Theorem 5.9 that

$$J(u^{\sigma^{\hat{\lambda}}}, \hat{\lambda}) = \inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}} J(u, \lambda),$$

but we also know that $J(u^{\sigma^{\hat{\lambda}}}, \lambda) = J(u^{\sigma^{\hat{\lambda}}}, \hat{\lambda})$ for any $\lambda \in \Lambda$. Then

$$\inf_{\lambda \in \Lambda} J(u^{\sigma^{\hat{\lambda}}}, \lambda) = \inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}} J(u, \lambda),$$

and so

$$\sup_{u \in \mathcal{U}} \inf_{\lambda \in \Lambda} J(u, \lambda) \geq \inf_{\lambda \in \Lambda} J(u^{\sigma^{\hat{\lambda}}}, \lambda) = \inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}} J(u, \lambda).$$

□

5.5 Limiting Arguments

We consider the limit $\varepsilon \to 0$ in order to recover the full embedding problem (OptSEP). Let $J^{\varepsilon}(\cdot, \cdot)$ be the objective function corresponding to taking some value ε and the corresponding h^{ε}. For each $\varepsilon > 0$ we seek optimisers λ^{ε} and τ^{ε} solving

- $F'(\tau^{\varepsilon}) = \dfrac{1}{2} \sum_k \lambda_k^{\varepsilon} h_{xx}^{\varepsilon}(W_{\tau^{\varepsilon}} - z_k),$
- $\mathbb{E}\left[h^{\varepsilon}(W_{\tau^{\varepsilon}} - z_j)\right] = -U_{\mu}(z_j), \quad \forall j,$

and therefore

$$\begin{aligned}
\inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}} J^{\varepsilon}(u, \lambda) &= J^{\varepsilon}(u^{\varepsilon}, \lambda^{\varepsilon}) \\
&= \mathbb{E}\left[F(\tau^{\varepsilon})\right] - \sum_k \lambda_k^{\varepsilon} \mathbb{E}\left[U_{\mu}(z_k) + h^{\varepsilon}(W_{\tau^{\varepsilon}} - z_k)\right] \\
&= \mathbb{E}\left[F(\tau^{\varepsilon})\right].
\end{aligned}$$

Also, for any k,

$$\begin{aligned}\left|\mathbb{E}\left[|W_{\tau^\varepsilon}-z_k|\right]+U_\mu(z_k)\right| &\le \left|\mathbb{E}\left[|W_{\tau^\varepsilon}-z_k|\right]-\mathbb{E}\left[h^\varepsilon(W_{\tau^\varepsilon}-z_k)\right]\right| \\ &\qquad\qquad\qquad\qquad + \left|\mathbb{E}\left[h^\varepsilon(W_{\tau^\varepsilon}-z_k)\right]+U_\mu(z_k)\right| \\ &\le \sup_x \left||x|-h^\varepsilon(x)\right| \\ &\to 0 \quad \text{as } \varepsilon\to 0.\end{aligned}$$

Note that this applies only at the atoms of μ and to infer that $\mathcal{L}(W_{\tau^\varepsilon})\to\mu$ we require this to hold at all x. We will not have convergence for all x since we have $\mathbb{P}\left(\tau^\varepsilon=T\right)>0$, and so τ^ε embeds mass outside of the support of μ along $\{t=T\}$. However, as we let $T\to\infty$ we will recover the full distribution.

We have shown that we can restrict ourselves to optimising over $\bar{\Lambda}=[\underline{\lambda},\bar{\lambda}]^n$, and this ensures that we have an optimising vector λ^ε, which converges to the zero vector in l^∞ as $\varepsilon\to 0$, since $\lim_{\varepsilon\to 0}\bar{\lambda}=0$ for any j.

For $x\notin\operatorname{supp}(\mu)$, by our choice of h^ε we have $\sum_k \lambda_k^\varepsilon h_{xx}^\varepsilon(x-z_k)\to 0$ as $\varepsilon\to 0$. For $z_j\in\operatorname{supp}(\mu)$,

$$\lim_{\varepsilon\to 0} h_{xx}^\varepsilon(z_j-z_k)=\begin{cases}0, & k\neq j\\ \infty, & k=j.\end{cases}$$

However, $0\le\lim_{\varepsilon\to 0}\sum_k\lambda_k^\varepsilon h_{xx}^\varepsilon(z_j-z_k)\le\lim_{\varepsilon\to 0}\bar{\lambda}_j^\varepsilon h_{xx}^\varepsilon(0)\le F'(0)$, and so the limit exists along some sequence. We can therefore find a sequence $(\varepsilon_n)_n$ such that the functions $h^n(x):=\sum_k\lambda^{\varepsilon_n}h_{xx}^{\varepsilon_n}(x-z_k)$ converge pointwise as $n\to\infty$. Our Root barrier stopping regions, $\left(\mathcal{D}^{\lambda^{\varepsilon_n}}\right)^\complement$ must then converge to some atomic Root stopping region $\mathcal{D}^\infty$ with hitting time τ^∞, and by Root [1969, Lemma 2.4], $(W_{\tau^{\varepsilon_n}},\tau^{\varepsilon_n})\xrightarrow{\mathbb{P}}(W_{\tau^\infty},\tau^\infty)$.

We can now consider also allowing $T\to\infty$. For each T we have some Root barrier with stopping time τ^T (corresponding to τ^∞ in the above for a fixed T) such that $\mathbb{E}\left[|W_{\tau^T}-z_j|\right]=U_\mu(z_j)$ for all j, however $\mathbb{E}\left[|W_{\tau^T}-x|\right]>U_\mu(x)$ for $x\notin\operatorname{supp}(\mu)$. As $T\to\infty$, these Root barriers will converge in the sense of Root [1969], and since the only stopping at $x\notin\operatorname{supp}(\mu)$ occurs at T, in the limiting barrier we will stop no mass at these points. The potential function will then be linear between the atoms of μ and agree with U_μ at the atoms, and therefore coincides with U_μ, so this barrier must embed μ.

Our results are summarised in the following.

Theorem 5.11. *Let $J^{\varepsilon,T}(\cdot,\cdot)$ be the objective function corresponding to taking some* *fixed $\varepsilon > 0$ and $T < \infty$. Then*

$$\inf_{\lambda\in\Lambda}\sup_{u\in\mathcal{U}} J^{\varepsilon,T}(u,\lambda) = \mathbb{E}\left[F(\tau^{\varepsilon,T})\right]$$

where $\tau^{\varepsilon,T}$ is the hitting time of a Root barrier where $\mathbb{E}\left[h^{\varepsilon}(W_{\tau^{\varepsilon}} - z)\right] = -U_{\mu}(z)$ for all atoms z of μ.

For any $T < \infty$, $\tau^{\varepsilon,T} \xrightarrow{\mathbb{P}} \tau^{T}$ along some subsequence as $\varepsilon \to 0$, where τ^{T} is the hitting time of a Root barrier and $W_{\tau^T} \sim \mu^T$, where $U_{\mu^T}(z) = U_{\mu}(z)$ for atoms z of μ.

As $T \to \infty$, we have $\tau^{T} \xrightarrow{\mathbb{P}} \tau$ along some subsequence, where τ is the hitting time of a Root barrier and $W_{\tau} \sim \mu$.

Since we know that there is a unique Root barrier embedding any distribution, and that the corresponding stopping time is an optimiser of (OptSEP) when we choose F concave, we have that τ in the Theorem 5.11 is an optimiser of (OptSEP). We can in fact deduce this without prior knowledge of the Root solution to the embedding problem, as shown in the following.

Corollary 5.12. *For a distribution μ with finitely many atoms, there is a Root barrier whose stopping time is an optimiser of* (OptSEP) *when $F \in C^1([0,T])$ is a concave, increasing function of time such that $\mathbb{E}\left[F(H_{x_*} \wedge H_{x^*})\right] < \infty$.*

Proof. The stopping times $\tau^{\varepsilon,T}$ converge in probability along some subsequence to a Root stopping time τ, and since F is continuous, $F(\tau^{\varepsilon,T}) \xrightarrow{\mathbb{P}} F(\tau)$ also. For any ε and T, $F(\tau^{\varepsilon,T}) \leq F(H_{x_*} \wedge H_{x^*})$, which is integrable, and we therefore have $\mathbb{E}\left[F(\tau^{\varepsilon,T})\right] \to \mathbb{E}\left[F(\tau)\right]$ along our subsequence.

Suppose there exists a uniformly integrable stopping time σ such that $W_{\sigma} \sim \mu$ and $\mathbb{E}\left[F(\sigma)\right] > \mathbb{E}\left[F(\tau)\right]$, and let $\sigma^T := \sigma \wedge T$ for any $T > 0$. Note that these stopping times are also dominated by $H_{x_*} \wedge H_{x^*}$ and so $\mathbb{E}\left[F(\sigma^T)\right] \to \mathbb{E}\left[F(\sigma)\right]$ as $T \to \infty$. Also if $W_{\sigma^T} \sim \mu^T$, then for any k,

$$\left|\mathbb{E}\left[h^{\varepsilon}(W_{\sigma^T} - z_k)\right] + U_{\mu}(z_k)\right| \leq \left|\mathbb{E}\left[h^{\varepsilon}(W_{\sigma^T} - z_k)\right] + U_{\mu^T}(z_k)\right| + \left|U_{\mu^T}(z_k) - U_{\mu}(z_k)\right| \to 0,$$

as $\varepsilon \to 0$, $T \to \infty$ and therefore

$$\lim_{T\to\infty}\lim_{\varepsilon\to 0}\left(\mathbb{E}\left[F(\sigma^T)\right] - \sum_k \lambda_k \left(U_{\mu}(z_k) + \mathbb{E}\left[h^{\varepsilon}(W_{\sigma^T} - z_k)\right]\right)\right) = \mathbb{E}\left[F(\sigma)\right].$$

We know that $U_\mu(z_k) + \mathbb{E}\left[h^\varepsilon(W_{\tau^{\varepsilon,T}} - z_k)\right] = 0$ for any k and therefore there exist small ε and large T such that

$$\mathbb{E}\left[F(\tau^{\varepsilon,T})\right] + \sum_k \lambda_k \left(U_\mu(z_k) + \mathbb{E}\left[h^\varepsilon(W_{\tau^{\varepsilon,T}} - z_k)\right]\right) < \\ \mathbb{E}\left[F(\sigma^T)\right] + \sum_k \lambda_k \left(U_\mu(z_k) + \mathbb{E}\left[h^\varepsilon(W_{\sigma^T} - z_k)\right]\right)$$

for any λ, so $J(\lambda, u^{\tau^{\varepsilon,T}}) < J(\lambda, u^{\sigma^T})$ for any λ, where J is a payoff function in (OptCon) with our chosen ε and T. However, there is some $\hat{\lambda}$ such that $J(\hat{\lambda}, u^{\tau^{\varepsilon,T}}) = \sup_{u \in \mathcal{U}^d} J(\hat{\lambda}, u)$, which is a contradiction, so no such σ exists. □

5.6 Interpretation of Adjoints as Dual Functions

In this section we relate quantities from the above calculations to the dual functions G and H that we have discussed in Section 1.3.3. For the Root barrier with boundary given by $R(x)$ and hitting time τ, the optimal dual functions are given in Cox and Wang [2013a] as

$$\begin{aligned} G(x,t) &:= \int_0^t M(x,s)\mathrm{d}s - Z(x), \\ H(x) &:= \int_0^{R(x)} \big(F'(s) - M(x,s)\big)\,\mathrm{d}s + Z(x), \end{aligned}$$

where Z is some convex function and $M(x,t) = \mathbb{E}^{x,t}\left[F'(\tau)\right]$. Note that the process $M(W_t, t) = \mathbb{E}\left[F'(\tau)\middle|\mathcal{F}_t\right]$ is a martingale, and since F is concave we have $M(x,t) \geq F'(t)$, with equality in the stopping region $\{t \geq R(x)\}$.

For given λ and ε, we have seen that the optimal stopping time, τ^λ, is such that

$$F'(\tau^\lambda) = \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_{\tau^\lambda} - z_k).$$

Since we have $u_t^\lambda = \mathbf{1}\{t < \tau^\lambda\}$, we see that

$$
\begin{aligned}
\frac{1}{2}P_t &:= -\frac{1}{2}\mathbb{E}\left[\sum_k \lambda_k h_{xx}^\varepsilon(X_T - z_k)\middle|\mathcal{F}_t\right] \\
&= -\frac{1}{2}\mathbb{E}\left[\sum_k \lambda_k h_{xx}^\varepsilon(W_{\tau^\lambda} - z_k)\middle|\mathcal{F}_t\right] \\
&= -\mathbb{E}\left[F'(\tau^\lambda)\middle|\mathcal{F}_t\right],
\end{aligned}
$$

and therefore

$$P_t = -2M(W_t, t).$$

Recall that for a given distribution μ there is a unique Root barrier whose hitting time embeds μ, and this stopping time is always optimal in the sense that it solves (OptSEP). We have seen that our embedding is determined by the correct choice of λ, and given this λ we chose to stop as soon as $F'(t) \leq \frac{1}{2}\sum_k \lambda_k h_{xx}^\varepsilon(W_t - z_k)$. This choice of the stopping time corresponds to choosing the Root barrier solution, and we need no further conditions for optimality.

5.7 The Rost and Cave Embeddings

In Cox and Wang [2013b], the authors replicate the optimality arguments of the Root barrier in Cox and Wang [2013a] for the Rost barrier. In this case the dual functions are very similar, and if we repeat our arguments for a convex, increasing function $F(t)$ then we expect to recover the unique Rost embedding and will have the same interpretation of the adjoint process P_t. Our optimal stopping time in this situation however will *not* be $\tau = \inf\{t \geq 0 : F'(t) \geq \frac{1}{2}\sum_k \lambda_k h_{xx}^\varepsilon(W_t - z_k)\}$, which is the obvious opposite of the Root stopping time.

For the Rost case consider a payoff function $F(t)$ which is convex, decreasing, and differentiable with $F(0) = 0$. Since the payoff is decreasing, if we do not have local time constraints then it is always optimal to stop the process immediately, and so for this problem we are required to bound the local time from below. Our problem is then

$$\inf_{\lambda\in\Lambda}\sup_{u\in\mathcal{U}^d}\mathbb{E}\left[F(T) - \int_0^T (1-u_t)^2 F'(t)\mathrm{d}t + \sum_k \lambda_k \left(U_\mu(z_k) + h^\varepsilon(X_T - z_k)\right)\right],$$

where

- $\mathcal{U}^d := \{u : [0,T] \times \Omega \to [0,1] \big| \, u \text{ is progressively measurable, decreasing, } u_T = 0\}$
- $\Lambda := \{(\lambda_1, \lambda_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : \; \lambda_k \geq 0 \, \forall k\}$
- $X_t = \displaystyle\int_0^t u_s \mathrm{d}W_s, \qquad \forall t \in [0,T], u_t \in \mathcal{U}^d.$

We can then formulate our stochastic maximum principle and, following the reasoning from the Root embedding, we expect the stopping time $\tau = \inf\{t \geq 0 : \; F'(t) \geq \frac{1}{2}\sum_k \lambda_k h^{\varepsilon}_{xx}(W_t - z_k)\}$. However, if we do not have any atoms near 0 then this would imply that it is optimal to stop immediately for any choice of λ, which we know to be incorrect. To counter this we can reformulate the problem to enforce the local time condition at all points, not just at atoms of μ.

Consider $\varepsilon > 0$ for which there is a set $\{z_0, z_1, \ldots, z_n\}$ where $z_0 = x_*$, $z_n = x^*$, and $z_{j+1} = z_j + \frac{\varepsilon}{2}$. Then we can define $\Lambda := \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \; \lambda_k \geq 0 \, \forall k\}$ and consider

$$\inf_{\lambda \in \Lambda} \sup_{u \in \mathcal{U}^d} \mathbb{E}\left[F(T) - \int_0^T (1-u_t)^2 F'(t)\mathrm{d}t + \sum_{k=0}^{n} \lambda_k \left(U_\mu(z_k) + h^{\varepsilon}(X_T - z_k)\right)\right].$$

In this framework we expect to find that for fixed λ the optimal control is $u_t = \mathbf{1}\{t \leq \tau\}$ for $\tau = \inf\{t \geq 0 : F'(t) \geq \frac{1}{2}\sum_k \lambda_k h^{\varepsilon}_{xx}(W_t - z_k)\}$, and then again that the optimisation over λ enforces the correct embedding condition. It is clear that if we enforce the local time condition at all x in the setup of the Root problem then we recover the same optimal control for any λ, and the optimal λ would still guarantee the correct embedding, so the solution would be unchanged.

If we can solve the problem in this form for the Root and Rost embeddings, we may hope to transfer the results over to the cave embedding. For the cave embedding problem our function F is convex, decreasing for $t \in [0, t_0]$, and concave, increasing for $t \geq t_0$. Since the payoff is no longer a sub or supermartingale, we will need to fix the local time condition rather than taking an inequality. This corresponds to taking unconstrained Lagrange multipliers λ_k. We expect the same Root and Rost stopping times in the regions $\{t > t_0\}$ and $\{t < t_0\}$ respectively, and again the embedding will be determined by the Lagrange multipliers, however, as we have seen previously, there will be an extra condition required to ensure this is the optimal embedding. The correct formulation of this problem should allow us to recover the condition (Γ).

We have seen (at least in the Root embedding case) that when the optimal control is determined by a stopping time σ, we have

$$J(u^\sigma, \lambda) = \mathbb{E}\left[\int_0^\sigma \left(F'(t) - \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k)\right) \mathrm{d}t\right] - \sum_k \lambda_k \left(U_\mu(z_k) - h^\varepsilon(W_0 - z_k)\right)$$

and that we should stop our process when the penalisation of accruing local time becomes larger than the gains from the payoff, i.e. when $F'(t) \leq \frac{1}{2}\sum_k \lambda_k h^\varepsilon_{xx}(W_t - z_k) = -\frac{1}{2}P_t$. In Section 5.6 we saw that $P_t = -2M(W_t, t)$, and we are therefore comparing $F'(t)$ and $M(W_t, t)$, which is also the case in (Γ). This approach to the cave problem highlights that the condition (Γ) is the equilibrium point in the trade-off between accruing local time and ensuring gains from the payoff function.

5.8 Conclusions

In this chapter we have translated (OptSEP') into a FBSDE-constrained optimisation problem, and then a stochastic optimal control problem. Although Skorokhod embedding problems have been linked to FBSDEs in the past, we believe that our methods provide a novel approach to (OptSEP), and that many interesting questions can be asked, and solved, in this new framework. The relation between (OptSEP) and optimal stopping problems allows us to use ideas such as the stochastic maximum principle and the dynamic programming principle to learn more about optimal embeddings.

We have given a full solution of the optimal control problem in the example of the Root embedding, and have shown in Theorem 5.11 that we recover a Root barrier stopping region in the limit. This allows us to reprove the existence of an optimal Root stopping time solution to (OptSEP) in Corollary 5.12. Section 5.7 gives ideas on how to construct (OptCon) in the cases of the Rost and cave embeddings, and with the correct formulation these methods could provide further insight into the condition (Γ).

Chapter 6

Conclusions and Further Work

In this thesis we have presented three alternative formulations of the following problem: given a probability distribution μ, a Brownian motion W and a payoff function F, find a stopping time τ that maximises $\mathbb{E}\left[F(W_\sigma, \sigma)\right]$ over stopping times σ such that $W_\sigma \sim \mu$ and $(W_{t\wedge\sigma})_{t\geq 0}$ is uniformly integrable.

In each of these formulations we are able to prove the existence of optimisers of this problem when we consider certain payoffs which are known to give optimisers with certain geometric properties. For example, in Chapter 5 we prove that, under certain conditions, when F is concave there is an optimiser which is the hitting time of a Root barrier. In particular, we do this without prior knowledge of the Root solution. Furthermore, these approaches allow us to prove properties of these optimisers without heavy probabilistic machinery. The discrete setup of Chapter 2 gives a simplified discrete version of the idea of Stop-Go pairs from Beiglböck et al. [2017b] and therefore allows us to prove geometric properties of the stopping regions.

The cave and K-cave embeddings are the first known examples of embeddings which are not uniquely determined by their geometric structure, and we have presented two approaches to finding the extra condition necessary for optimality. This condition is equivalent to the existence of optimisers in the dual superhedging problem, and we have proved duality results in both discrete and continuous time. In particular we have shown that dual optimisers always exist in some weighted space where we allow the dual feasible variables to grow exponentially in time. In the primal problem this corresponds to the exponential decay of the unstopped mass. Technically, we require this weighting to ensure the existence of interior points of the primal feasible set, but we have seen that this exponential decay appears to be a necessary condition in the example of the Rost embedding, for example.

We have seen that the original Skorokhod embedding problem has many natural extensions, and these can be interpreted in the formulations of the problem given in this thesis. The results of Yong and Zhou [1999] allow us to consider controlled processes of the form $\mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t + \sigma(t, X_t, u_t)\mathrm{d}W_t$ and in this framework it should therefore be simple to consider more general diffusions in place of Brownian motion in (OptSEP). Similarly we can consider a diffusion by adjusting the heat operator in Chapter 4, or its discrete equivalent in Chapter 2. In these cases we still require the exponential decay in order to show duality, and so the volatility of the process will have to be considered in this.

As discussed previously, one adjustment of (SEP) is to consider the multi-marginal version, and we have given comments in the case of the K-cave embedding in Section 3.8. It should be possible to consider the more general multi-marginal problem in the framework of Chapter 2. If we consider the n-marginal problem then we have vectors $(p^i_{j,t})_{j,t}$ for each $i = 1, \ldots, n$, and ultimately these methods could lead to interesting duality results for the multi-marginal problem. Similarly we can consider n-dimensional control processes in Chapter 5.

Finally, we have restricted ourselves to embedding problems which we know give hitting time solutions for (W_t, t), but other embeddings take the form of hitting times for different processes. The Azéma-Yor solution is the hitting time for (W_t, M_t) where $M_t := \sup\{W_s : s < t\}$. It may be possible to adjust the formulations in this thesis to consider embeddings such as the Azéma-Yor embedding. In Chapter 5 we make extensive use of the local time of the process in order to ensure we have the correct embedding, and for this reason it seems likely that the problem can be reformulated to consider the Vallois solution of (SEP).

We have studied the two-sided barrier versions of the Root and Rost embeddings, and with the methods introduced in this thesis it may be possible to examine similar extensions of other embeddings, for example the Azéma-Yor and Vallois embeddings. The Azéma-Yor solution is known to maximise the law of the supremum process $(M_t)_t$ in some sense, and there is a similar inverse embedding which minimises the law of the supremum, see for example Perkins [1986] and Hobson and Pedersen [2002]. Both of these solutions are hitting times for (W_t, M_t) and so it may be possible to combine these problems in the same way that the Root and Rost barriers are used to find the cave embedding. For example, fix some $k_0 > 0$ and let $F : [0, \infty) \to \mathbb{R}$ be decreasing on

$[0, k_0)$ and increasing on $[k_0, \infty)$. We expect that there is a solution of $\sup_\tau \mathbb{E}[F(M_\tau)]$, where the supremum is taken over stopping times τ that satisfy (SEP). Furthermore, we expect that this solution will be a hitting time for (W_t, M_t) where the stopping region looks like the Perkin's solution and the Azéma-Yor solution separated by $\{M_t = k_0\}$. If such an embedding can be found, it will likely feature the same non-uniqueness property as the cave and K-cave barriers, and would require an extra condition for optimality. It would be interesting to see if this condition has any relation to (Γ).

In summary, we have presented an array of techniques applicable to proving duality results in optimal Skorokhod embedding problems. These can be used to recover previous embeddings, prove the existence of dual optimisers in certain spaces, and are also robust in that they can be easily manipulated to consider generalisations of the classical embedding problem.

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